Exercises of *The Mathematica GuideBook for Graphics*

CHAPTER 1

Exercises

1.[L2] Game of Life, Growth Process, Langton's Ant

a) Implement the "Game of Life" (see, for instance, [36✱]) for a square grid. At the $(i+1)$st step, a grid point remains alive, provided the number of its living neighbors at the ith step is one of several prescribed values. A dead grid point comes back to life in the $(i+1)$st step, provided the number of its living neighbors is one of several prescribed values. Put the prescribed numbers (i.e., the rules of the game) in the lists `RemainsLiving` and `ComeAlive`. Display the resulting configuration.

b) Model the following growth process on a square grid [4✱]: If grid points have been occupied by generations 1 to $i-1$, the ith generation is determined as follows: For every grid point of the generation i, two of the four neighbors are chosen at random. If these two neighbors are already occupied by any of the earlier generations, they will not be occupied again. If a grid point is chosen more than one time in the ith generation, it is occupied only starting from one of the grid points of the $i-1$st generation. The start generation is just one grid point.

Visualize the growth process. Program the growth process in a functional style.

c) Model the movement of "Langton's ant" [7✱], [6✱], [33✱], [18✱]. It moves on a square lattice with white and black sites. Initially all sites are white. At each time step the ant makes a step of length 1. As it enters a site, the site changes to the opposite color and the ant changes its moving direction: it will move right after entering a black site and it will move left after entering a white site.

2.[L3] Brillouin Zones

Create a graphic of the first few Brillouin zones for a 2D square grid. For a discussion of Brillouin zones, see any textbook on solid state physics, for example, [21✱], [53✱], [26✱], [27✱], [28✱], and [2✱]; for a more mathematical discussion, see [47✱], [5✱], [25✱], and [29✱].

For a straightforward implementation of Brillouin zones, we need to carry out the following steps:

- Create a regular grid for which the Brillouin zones are desired.
- Choose a fixed grid point (e.g., {0, 0}), and construct the perpendicular bisectors for the lines between this point and all of

its neighbors (up to some maximal distance).
■ The first Brillouin zone is the smallest domain that is bounded by the bisectors around the selected grid point.
■ The $i+1$st Brillouin zone is the smallest (possibly just a single point) connected domain that completely encloses the ith Brillouin zone (except for isolated points).
Construct the first nine Brillouin zones for a hexagonal grid.

3.[L1] Color Triangle

There is a dilemma in using `RGBColor[`*redPart*`,`*greenPart*`,`*bluePart*`]` for displaying "all" colors on a 2D region. It involves three independent parameters, but we can display the colors on at most a 2D surface. The Maxwell-Helmholtz color triangle (see, e.g., [30✱], [50✱]) offers one alternative. It is an equilateral triangle in which the barycentric coordinates of a point are interpreted as {*redPart*, *greenPart*, *bluePart*}, where the sum of the three parts is 1. Display the Maxwell-Helmholtz color triangle.

4.[L1] Conformal Mappings

Program a function
`ConformalMap[`*f*(*z*)`, {`*xmin*`, `*xmax*`}, {`*ymin*`, `*ymax*`}, PlotPoints -> {`*ppx*`, `*ppy*`}]`
that displays the following two images side by side:
■ A grid consisting of *ppx* horizontal and *ppy* vertical lines in the complex z-plane ($xmin \leq \mathrm{Re}(z) \leq xmax$, $y\,min \leq \mathrm{Im}(z) \leq y\,max$)
■ The image in the complex w-plane of this grid, obtained by applying the function f (i.e., $w = f(z)$)
Use the function `ConformalMap` to visualize various complex functions $f(z)$.

5.[L1] Cornet Isogons, Jarník Polygons

a) Program the display of Cornet isogons. They are, in general, self-intersecting n-gons with the following two properties:
■ The successive sides have lengths 1, 2, 3, …
■ The angles between any successive sides are the same for all vertices (without taking account of orientation)
Here is the construction (for details, see [40✱] and [39✱]). Suppose the prescribed angle α between sides is a rational fraction of 2π (to get reasonable computation times, the denominator should not be too large). Let d be the denominator in the ratio $(\pi-\alpha)/(2\pi)$. To get a Cornet isogon of order o, connect line segments of lengths 1, 2, 3, 4, … so that at successive vertices the angles are $(o\,d-1)$ times α, 1 times $-\alpha$, $(o\,d-1)\,\alpha$, and 1 times α…. After exactly $d^2/2$ (d even) or d^2 (d odd) steps, we get back to the starting point.

b) Write a one-liner that calculates the Jarník polygon of order o [23✱], [22✱]. The Jarník polygon of order o is the polygon whose edges are the counter-clockwise sorted vectors of the form $\{a, b\}$, $-o \leq a, b \leq o$, $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$. Visualize how the scaled Jarník polygon (scaled to fit in the unit circle) approach the curve $3/4\,x^2 - 1$, $-2/3 \leq x \leq 2/3$ and its images under rotations by $\pi/2$ [32✱].

6.[L1] Plot Approximation

a) Explain in detail how the following function works.

```
DistributionOfBends[image_Graphics] :=
Show[
Graphics[{{Thickness[0.01], Line[#]},
          {Thickness[0.001], Line[{#, {#[[1]], 0}}]& /@ #}}&[
            Thread[({#, Range[Length[#]]}& /@
             {Sort[N[360/(2Pi)]ArcCos[#[[1]].#[[2]]]&  /@
                 Partition[#/Sqrt[#.#]& /@
```

```
                  (Apply[Subtract, #]& /@ Partition[image[[1, 1,
                       1, 1]], 2, 1]), 2, 1]]})[[1]]]]],
      PlotRange -> All, Axes -> True,
      AxesLabel -> {"φ", "n"}, AxesOrigin -> {0, 0}];
```

Implement a routine `myPlot` that emulates the function `Plot`, especially the adaptive subdivision, for simple functions $f(x)$ (real-valued over the plot region). Compare

`DistributionOfBends[Plot[Sin[x] + Sin[3x]/3, MaxBend -> 2]`

with

`DistributionOfBends[myPlot[Sin[x] + Sin[3x]/3, MaxBend -> 2]`.

In a similar way implement a routine `myParametricPlot` that emulates `ParametricPlot` and compare

`ParametricPlot[{Cos[5x], Sin[3x]}, {x, 0, 2Pi}]`

with

`myParametricPlot[{Cos[5x], Sin[3x]}, {x, 0, 2Pi}]`.

b) Create a visualization to show that the plots generated by `Plot` and `ParametricPlot` consist of piecewise linear curves.

7.[L1] Rainbows

Model the result of a ray of white light, which is reflected inside a (round) drop of water. The refractive index for red and blue light are around 1.3331 and 1.343, respectively. Make a plot. In view of symmetry, it suffices to look at a circular segment of the spherical drop.

8.[L1] Warped Patterns

Cover part of the plane with an area-filling pattern, and then warp it.

9.[L1] Moiré Pattern

Construct a Moiré pattern.

10.[L1] Star and *Mathematica*, Touching Circles, Triptych Fractal and, Sum of Digits, and Animations

Without running the following five programs, predict what one will get.

a)

```
Show[Graphics[
MapIndexed[{Hue[Random[]],
            Text[StyleForm[#1, FontFamily -> {"Helvetica", "Courier",
                 "Times"}[[Random[Integer, {1, 3}]]], FontSize -> 8],
                    {Subtract @@ #2, 10 - #2[[1]]}]}&,
           Join[#, Rest[Reverse[#]]]& /@
  Table[{"M", "A", "T", "H", "E", "M", "A", "T", "I", "C", "A"}[[i]],
        {j, 11}, {i, j}], {2}]]];
```

b)

```
Star[n_Integer?Positive] :=
Show[Graphics[Line/@ (Distribute[{#, #}, List]&[
                      Table[{Cos[i 2Pi/n], Sin[i 2Pi/n]}, {i, n}]])],
     PlotRange -> All, AspectRatio -> Automatic]

Star[17]
```

c)

```
Show[Graphics[{Thickness[0.001],
     Function[r, Line[Transpose[{r, #}]]& /@
      Drop[Transpose[FoldList[Plus, 0,
        NestList[-2#(1 - #^2)(1 + #)&, #, 20]]& /@ r], 2]][
                                       Range[0., 1., 0.002]]}],
      Frame -> True, PlotRange -> All, FrameTicks -> None];
```

d)

```
With[{pi = N[Pi]},
Function[pointList,
Show[Graphics[Function[piece,
MapIndexed[{GrayLevel[If[EvenQ[#2[[1]]], 0, 1/2]],
            Polygon[Join[#1[[1]], Reverse[#1[[2]]]]]}&,
Partition[Transpose[Sort[#, OrderedQ[{#1[[2]], #2[[2]]}]&]& /@
                 Take[pointList, piece]], 2, 1]]] /@
  Apply[Flatten[{##}]&, Partition[Position[Apply[SameQ, Partition[
Function[l, Flatten[Position[l, #]& /@ Sort[l]]] /@
    Map[Last, pointList, {-2}], 2, 1], {1}], False], 2, 1], {1}]],
        AspectRatio -> 1/4]][
     Table[{φ, Sin[φ + d]}, {φ, 0, 8pi, pi/400}, {d, 0, pi, pi/6}]]];
```

e)

```
Show[Graphics[{Thickness[0.001], MapIndexed[{Hue[#2[[1]]/120], #1}&,
  Line[MapIndexed[{#2[[1]], #1}&, #]]& /@
    (Function[base, Rest[FoldList[#1 + (Plus @@
         #2)&, 0, IntegerDigits[Range[100], base]]]] /@ Range[2, 120])]}],
     Frame -> True, PlotRange -> All];
```

f) Given a circle, place n circles along its boundary in such a manner that these circles touch each other at the same point where they intersect with the first circle. Iterate this process.

g) Make an animation of $f(x+a)+f(x-a)$ as a function of a (where $f(x)$ is a smooth positive function with a single maximum) that gives three maxima for certain values of a.

h) Take a reflection-symmetric pentagon with equal side length and iteratively reflect the pentagon on its edges. Make an animation showing how the set of reflected pentagons varies as the shape of the initial pentagon changes. Do the same with a decagon.

i) Make an animation showing the reflection of a set of rays coming from the origin in a closed, smooth, randomly deforming plane figure.

j) Write a one-liner that visualizes the formation of all possible partial sums of $\sum_{k=0}^{n} e^{i n \varphi}$.

11.[L1] Random Lissajous Figures, Joining Curves Smoothly

a) Program the generation and visualization of Lissajous figures (2π-periodic sums of trigonometric functions) that have some randomness. The resulting curves should exhibit a certain symmetry.

b) Given a list of smooth nonclosed curves in the form of `Line` primitives, write a program so that by translating and rotating, the curves join them together in such a way that the resulting curve is also a smooth one. Test the program with some random selection of curve pieces.

12.[L1] Walsh Functions, Sorting Game, Ball Moves, Rectangle Packings

a) Program a Walsh function. For given

$$n = i_0\, 2^0 + i_1\, 2^1 + i_2\, 2^2 + \cdots + i_n\, 2^n$$

a possible representation for Walsh function is

$$\text{Walsh}\,(x) = \text{sign}\left(\sin(2\,\pi\,x)^{i_0} \prod_{k=1}^{n} \cos(2^k\,\pi\,x)^{i_k}\right)$$

for $0 < x < 1$, and the definition extends periodically. Plot the first few Walsh functions along with the associated trigonometric functions obtained from the above formula. For more on Walsh functions, see [43✱], [41✱], and [24✱].

b) Consider the following "game" [13✱], [14✱]: The numbers 1 to n are distributed among the lattice points $\{1\}$, $\{2\}$, …, $\{n\}$. At each step, the largest number k having numbers greater than k to its right is selected and placed at the smallest empty lattice point such that no numbers greater than k are to its right. This process is repeated until it naturally ends.

Example:{3,2,5,1,4,…}→{3,2,□,1,4,5,…}→{□,2,□,1,4,5,3,…}→{□,2,□,1,4,□,3,5,…}→{□,2,□,1,□,□,3,5,4,…}→→ {□,2,□,1,□,□,3,□,4,5,…}→{□,□,□,1,2,□,3,□,4,5,…}

Implement this "game" and calculate and visualize the final positions for some random initial configurations for the first 100 numbers. Play the "game" for all initial configurations of 1, …, 8.

c) Consider the following ball system [52✱], [45✱]: The numbered balls 1 to n are distributed among the lattice points $\{1\}$, $\{2\}$, …, $\{m\}$ ($m \geq n$). At each step, all numbers are shifted to the right according to the following rules:

- the left-most ball is moved to the first empty lattice point to its right
- the next left-most ball is moved to the first empty lattice point to its right and so on
- each ball is moved only once
- the lattice points are cyclically continued (meaning $\{m + j\} \equiv \{j\}$.

Example for $n = 6$, $m = 12$:

{□, □, □, 1, 2, □, 3, □, 4, 5, 6, □} → {□, □, □, 2, 1, □, 3, □, 4, 5, 6, □} →
{□, □, □, □, 1, 2, 3, □, 4, 5, 6, □} → {□, □, □, □, 1, 2, □, 3, 4, 5, 6, □} →
{□, □, □, □, 1, 2, □, 3, □, 5, 6, 4} → {5, □, □, □, 1, 2, □, 3, □, □, 6, 4} →
{5, 6, □, □, 1, 2, 0, 3, □, □, □, 4}

This process is repeated k times.

Implement this ball system and visualize how the numbered balls move for $m = 500$ and 500 steps.

d) Implement the following packing of rectangles [10✱]: Fix an aspect ratio and randomly place a rectangle inside a given square (use a random center and a random orientation). Now, let this rectangle expand in size (while keeping its aspect ratio fixed) until one of its edges hits the bounding square. Randomly place a second rectangle inside the bounding square and outside the first rectangle and let it expand until one of its vertices or edges either hits the bounding square or the first rectangle. Continue this process until n rectangles have been placed. Visualize the resulting packings for various aspect ratios of the placed rectangles.

13.^{L2} L-Systems with Rounded Corners, Billiard Reflections

a) Given a list of positive real numbers and a set of lists, each of length 2 and consisting of a positive real number r and a positive or negative real number $\delta\varphi$, interpret the numbers as lengths of line segments in the plane, and the lists as circular arcs of radius r formed by turning either left or right by an angle $\delta\varphi$. Construct a corresponding smooth connected curve by connecting the curve segments in the order described by the lists. Examine the resulting curves for various lists.

b) Implement a function that shows multiple, billiard-type reflections on a regular $n$gon. Show a bouncing ball inside various $n$gons. Show various “qualitatively different” billard paths inside a pentagram. Consider what happens when the polygon corners are cut off so that n (small) holes are present [35✱], [37✱].

14.^{L1} Rough Functions, Random Walks

a) Explain the operation of the following five functions: `mima`, `mPeano`, `summer`, `minkowski`, and `digits`. Plot them using either `Plot` or `ParametricPlot`, respectively.

The function `mima`:

```
mima[0.0] = 0.0;
mima[1.0] = 1.0;
mima[x_?NumberQ] := mima[x] =
If[EvenQ[Length[DownValues[mima]]],
   Min[#[[2]]] + (Max[#[[2]]] - Min[#[[2]]])*
       (x - #[[1, 1]])/(#[[1, 2]] - #[[1, 1]])/3,
   Min[#[[2]]] - (Max[#[[2]]] - Min[#[[2]]])*
       (x - #[[1, 1]])/(#[[1, 2]] - #[[1, 1]])/3]&[{#, mima /@ #}&[
{Max[Select[#, Function[t, x - t > 0]]],
 Min[Select[#, Function[t, x - t < 0]]]}&[
         Flatten[Level[#, {-1}]& /@ ((Hold @@ #)& /@
                (First /@ Drop[DownValues[mima], -1]))]]]]
```

The function `mPeano`:

```
mPeano[x_Real] :=
ToExpression[StringJoin["0.", ToString /@ #]]& /@ {Rest[#[[1]]],
     #[[2]]}&[Transpose[Partition[Prepend[RealDigits[x][[1]], 0], 2]]]
```

The function `summer`:

```
summer[x_Real] := Plus @@ Flatten[RealDigits[x]]
```

The function `minki` (a far relative to a function from [48✱], [38✱]).

```
minki[x_Real?(0 < # < 1&)] :=
 Plus @@ MapIndexed[-(-1)^#2[[1]] 2.^-#1&,
   Rest[FoldList[Plus, 0, RotateLeft[First[#], Last[#]]&[
                                  RealDigits[x]]]] - 1]
```

The function `digits`:

```
digits[x_Real, {n_Integer, m_Integer}] :=
{#[[n]], #[[m]]}&[Flatten[Append[
 #, Table[0, {16 - Length[#]}]]&[RealDigits[x][[1]]]]]
```

b) What picture results from the following?

```
f[x__] := If[EvenQ[Last[Date[]]], f[x, 1], Length[{x}]];
ListPlot[Array[f, 10000], PlotJoined -> True, PlotRange -> All]
```

c) Describe what the following input does.

```
sitesVisited[n_] :=
Module[{c, p = 0},
       Do[p = p + 2 Random[Integer] - 1;
          If[Head[c[p]] === c, c[p] = 1,
             c[p] = c[p] + 1], {n}];
       {#[[1, 1, 1]], #[[2]]}& /@ DownValues[c]]

ListPlot[sitesVisited[10^5],
         Frame -> True, Axes -> False, PlotRange -> All];
```

d) Describe what the following input does.

```
walkStep[l_List, {L_, δ_}] :=
Block[{λ = (# + 2 Random[Integer] - 1)& /@ l, p,
       𝕣 := Random[Integer, {1, δ}]},
 p = Partition[Append[λ, First[λ] + L], 2, 1];
 Which[# > L, # - L, # < 0, L + #, True, #]& /@
      Union[Table[
       Which[p[[k, 1]] >= p[[k, 2]], p[[k, 2]],
             p[[k, 2]] - p[[k, 1]] > δ,
             Sequence @@ {p[[k, 1]], p[[k, 1]] + 𝕣},
             True, p[[k, 1]]], {k, Length[p]}]]]

walkers[{L_, δL_}, steps_] :=
Block[{l = Table[δL k + 1, {k, 0, (L + 1)/δL - 1}]},
      Table[l = walkStep[l, {L, 2 δL}]; {#, k}& /@ l, {k, steps}]];

Show[Graphics[{PointSize[0.002],
               Map[Point, walkers[{999, 50}, 20000], {2}]}],
     Frame -> True, PlotRange -> All, AspectRatio -> 1];
```

e) An order o Sierpinski triangle-like structure on a square grid can be generated by starting with the 3×3 matrix

$$m = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $(o - 1)$ times recursively replacing the 1s in the matrix by ms and the 0s by 3×3 0-matrices and flattening the resulting block matrices [1✱], [42✱], [9✱]. Model a random walk on such a Sierpinski triangle-like structure. For 1000 walks of length 1000 steps on a 4-Sierpinski triangle-like structure visualize how the square of the mean distance of the walker from the starting point depends on the number of steps. Visualize too how the number of different sites visited depends on the number of steps. Compare with a random walk on a quadrant of a square grid.

15.[L3] Random Figures, Deformed Lissajous Figures Animation, Voronoi Animation, Lévy Flights

a) Write a program to plot random figures with smooth boundaries on top of each other. To avoid large data sets with `Polygon[{`*manyCoordinates*`}]`, build the random figures using `Disk`.

b) Make an animation that shows a square array of varying smooth figures that at all times touch its four neighbors tangentially.

c) Describe what happens in the following animation.

```
g[z_, Z_, r_, τ_] := z + τ/25(z - Z)Exp[-Abs[(z - Z)/0.8]^2]*
                                       Exp[I τ/12 r Abs[(z - Z)]]

f[z_, τ_] = Sum[g[z, Random[Complex, {-1 - I, 1 + I}],
                     Random[Real, {-1, 1}], τ], {50}]/50;

Do[With[{C = {Re[#], Im[#]}& /@ Map[f[#, τ]&, curve, {-1}]},
   Show[Graphics[{Polygon[C], Thickness[0.006],
               MapIndexed[{Hue[(#2[[1]] + 50 τ)/2500], Line[#]}&,
               Partition[C, 2, 1]]}],
     PlotRange -> 1.5 {{-1, 1}, {-1, 1}}, Frame -> True,
     AspectRatio -> Automatic, FrameTicks -> None]],
   {τ, 1, 200, 199/60}];
```

d) Take n points that move along smooth random curves. Make an animation that shows how the inner cells of the Voronoi diagram [12✱], [34✱] of these points change. (The package ``DiscreteMath`ComputationalGeometry` `` implements the function `VoronoiDiagram` that calculates the Voronoi diagram for a set of points.)

e) The distribution function (Lévy distribution) $p_\beta(r)$ for a step (flight) of length r is $p_\beta(r) = (\beta - 1)\,(r+1)^{-\beta}$ [20✱], [31✱], [51✱], [19✱], [49✱]. Make an animation of a random walk in the plane where the step size obeys the distribution $p_\beta(r)$ and the directions are uniformly distributed. Let β vary from frame to frame.

16.[L2] Random Expressions, Random Subexpression Replacements, Perpetuities

a) This problem does not deal with 2D graphics, but with the command `Random` introduced in Subsection 1.5.6. Using the function `Random`, implement a function `randomExpression` that generates syntactically correct (but in general meaningless) *Mathematica* expressions of various complexities. An example might be:

```
Sin[Plot[Tan[1, 3, s], {1, 2, 3, Level[#, $]}], qw > jkl]
```

How does *Mathematica* react to the input of such expressions?

b) Starting from a "random" *Mathematica* expression, repeatedly select randomly a part of it and replace it with another randomly chosen part of the original expression. Analyze experimentally the average depth and leaf count of the resulting expressions as a function of the number of replacement iterations. As two examples for "random" expressions, choose for instance `Table[i j k, {i, 3}, {j, 3}, {k, 3}]` and

$$\sin(1) + \sqrt{\left(1+\pi^{2\log(\sqrt{3}+2)}\right)^{-1}} + \sqrt{\sqrt{\sqrt{e}+2}+1}\,.$$

c) Consider functions $f_k^{(n)}$ from the set of integers $\mathcal{S}_n = \{1, 2, \ldots, n\}$ to itself. Starting with $\mathcal{S}_n$ and repeatedly applying randomly selected $f_{k_j}^{(n)}$ soon yields a list with identical elements [11✱], [8✱]. Conjecture the asymptotic complexity of how often one has to apply the $f_{k_j}^{(n)}$ to obtain a list of identical elements as $n \to \infty$.

d) What does the following function calculate?

```
p[] := NestWhileList[Function[x, {x[[1]] + #, #}&[x[[2]] Random[]]],
                     {1., 1.}, (#1[[1]] != #2[[1]])&, 2][[-1, 1]]
```

17.^{L1} Iterated Random Numbers, Random Potential

a) The iterated random-number problem (presented by D. Withoff at the *Mathematica* Developer Conference in 1993) is as follows. Start with a prescribed integer n. Choose random numbers j_1 in the interval $(0, n)$ (using `Random[Integer, n]`) until one gets one that is less than or equal to n. Store the number k_{j_1} of calls required in a list of the form `{{n, ` k_{j_1} `}}`. Repeat this process until one gets a number $k_{j_2} < k_{j_1}$, and then add `{` j_1 `, ` k_{j_2} `}` to the list `{{n, ` k_{j_1} `}}`. Continue until $j_m = 0$. For example, the integer 23 might lead to the sequence 23, 23, 21, 12, 13, 17, 23, 12, 12, 5, 8, 19, 2, 1, 20, 16, 0. In this case, the result of `IteratedRandomNumbers[23]` is {`{23, 2}, {21, 1}, {12, 6}, {5, 3}, {2, 1}, {1, 3}`}.
Implement two solutions to this problem: one that contains pattern matching and replacement rules in the sense of Subsection 5.3.4 of the Programming volume [46✱] of the *GuideBooks*, and one that is based on a functional programming style.

b) For any sequence of positive (random) numbers $\{\beta\} = \{\beta_0, \beta_1, \ldots, \beta_n\}$, the function [44✱], [3✱]

$$\Psi_{\alpha,\{\beta\}}^{(0)}(x) = \sum_{k=0}^{n} \chi_k(x)\,(-1)^k \cos\!\left(\frac{\beta_k}{2}\,(x-l_k)\right) \exp\!\left(-\frac{\alpha}{\beta_k^{\,2}}\,\sin^2\!\left(\frac{\beta_k}{2}\,(x-l_k)\right)\right)$$

where $l_{k+1} = l_k + 2\pi/\beta_k$, $l_0 = 0$, and $\chi_k(x)$ is the characteristic function of the kth cell $\chi_k(x) = \theta(x - l_k)\,\theta(l_{k+1} - x)$ is a zero-energy solution to the Schrödinger equation

$$-\frac{1}{2}\,\frac{\partial^2 \Psi_{\alpha,\{\beta\}}^{(0)}(x)}{\partial x^2} + V_{\alpha,\{\beta\}}(x)\,\Psi_{\alpha,\{\beta\}}^{(0)}(x) = 0\,\Psi_{\alpha,\{\beta\}}^{(0)}(x)$$

with potential

$$V_{\alpha,\{\beta\}}(x) = \sum_{k=0}^{n} \frac{\chi_k(x)}{16}\left(\frac{\alpha}{\beta_k^{\,2}}\,(4\,\beta_k^{\,2} + \alpha) - 2\,\beta_k^{\,2} - \frac{\alpha^2}{\beta_k^{\,2}}\cos(2\,\beta_k\,(x-l_k)) - 8\,\alpha\cos(\beta_k\,(x-l_k))\right).$$

Here α is an arbitrary positive parameter.

Show that the $\Psi_{\alpha,\{\beta\}}^{(0)}(x)$ fulfill the Schrödinger equation and visualize $\Psi_{\alpha,\{\beta\}}^{(0)}(x)$ and $V_{\alpha,\{\beta\}}(x)$ for some (random) example sequences $\{\beta\}$.

18.^{L2} `Plot[…, …]` versus `Plot[Evaluate[…], …]`

For the following `Plot` inputs, discuss what happens when `Evaluate` gets wrapped around the first or second argument. Discuss the possibly different resulting graphics and the possibly different timings.

a) `Plot[Expand[10^20 (x - Exp[Log[x]])^10], {x, 0, 1}]`

b) `Plot[Expand[10^20 (x - Log[Exp[x]])^10], {x, 0, 1}]`

c) `Plot[Sum[x^k, {k, 10^4}], {x, 0, 1}]`

d) `Plot[Sum[x^k, {k, 10^20}], {x, 0, 1}]`

e)
```
f[x_] := Sin[x^2]^3 Exp[-Log[x^3 + 1] - x] ArcSin[x/3]^3;
Plot[D[f[x], x], {x, 0, 1}]
```

f)
```
f[x_] := Sin[x^2]^3 Exp[-Log[x^3 + 1] - x] ArcSin[x/3]^3;
Plot[f'[x], {x, 0, 1}]
```

g) `Plot[({x, x^2} + {Sin[x], x, Cos[x]})[[2]], {x, 0, 1}]`

h) `Plot[Block[{x = Sqrt[x]}, x^2], {x, 0, 1}]`

i) `c = 1; Plot[y = x + c; y, {x, 0, 1}]`

j) `Plot[Sin[x], {x + 0, 0, 1}]`

k) `Plot[Sin[x], {x, 0, 1, Sequence @@ {}}]`

l)
```
Plot[Log[-1 + 10^-100] - (10^1 x - 10^1 Exp[Log[x]] + 1) I Pi + x,
     {x, 0, 1}]
```

m) `Plot[Length[Append[Table[1, {100 x}], {1, 2}]], {x, 0, 1}]`

o) `Plot[{x, x^2} + Sin[x], {x, 0, 1}]`

p) `Plot[Integrate[Cos[y]^2, {y, 0, x}], {x, 0, 1}]`

q) `Plot[Sin[x], {x + C, 0, 2} - {C, 0, 1}]`

r)
```
Plot[Nest[Sqrt[Log[Exp[ArcSin[Sin[ArcCos[Cos[#]]]]]]^2]&, x, 10^2] - x,
      {x, 0, 1}]
```

19.[L2] Nomogram for Quadratic Equation

Construct and display a nomogram for the real roots of the quadratic equation $x^2 + a\,x + b = 0$ (a, b real). Solve the specific example $x^2 + 3\,x - 6 = 0$ with the nomogram constructed.

20.[L2] Black and White

Color the various regions (with the exception of the middle region) of the following picture by alternating black and white.

```
Show[Graphics[
With[{n = 23},
 Circle[Plus @@ #/2, Cos[2Pi/n]]& /@ Partition[
  Table[{Cos[φ], Sin[φ]},
        {φ, Pi/2 - 2Pi/(2n), 2Pi + Pi/2 - 2Pi/(2n), 2Pi/n}], 2, 1]],
    AspectRatio -> Automatic, PlotRange -> All]];
```

21.[L2] Connected Truchet Curves

The following subparts refer to the hexagonal Truchet pattern from Subsection 1.5.6.

a) Group the lines and circle pieces into such groups, so that each group forms a connected curve.

b) Create concentric, randomly colored "bands" along these groups.

c) Color the resulting curves in a rainbow-like fashion along the curves.

22.[L2] Connected Clusters on Square Lattice, Aperiodic Triangle Tilings

a) Given a set of line segments connecting lattice points on a square lattice, write a function that groups the lines into clusters. (Line segments belong to a cluster if they form a connected graph.)

b) A set of intersecting lines in 2D can induce an aperiodic tiling of the plane in the following way: Potentially, a subset $\mathcal{S}$ of the triangles formed by the intersecting lines can be subdivided into smaller triangles, congruent to other triangles from $\mathcal{S}$. Viewing these subdivisions as substitution rules and iterating them, yields a tiling of the plane. Find "proper" tilings (meaning all triangle edges start and end at vertices, rather than at the interior of other edges) for the following two sets of lines (from [15✱], [16✱], and [17✱]).

$$\begin{array}{ll}
y = 0 & y = 0 \\
y = \tan(1\,\frac{2\pi}{9})\,(x - \sigma_2) & y = \tan(1\,\frac{\pi}{12})\,(x - \tilde{\sigma}_5) = 0 \\
y = \tan(2\,\frac{2\pi}{9})\,x & y = \tan(2\,\frac{\pi}{12})\,(x - \tilde{\sigma}_4) = 0 \\
y = \tan(3\,\frac{2\pi}{9})\,(x - \sigma_1) & y = \tan(3\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_3) = 0 \\
y = \tan(4\,\frac{2\pi}{9})\,(x - \sigma_3) & y = \tan(4\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_2) = 0 \\
y = \tan(5\,\frac{2\pi}{9})\,(x - \sigma_3) & y = \tan(5\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_1) = 0 \\
y = \tan(6\,\frac{2\pi}{9})\,(x - \sigma_1) & x = 0 \\
y = \tan(7\,\frac{2\pi}{9})\,x & y = \tan(7\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_1) = 0 \\
y = \tan(8\,\frac{2\pi}{9})\,(x - \sigma_2) & y = \tan(8\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_2) = 0 \\
 & y = \tan(9\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_3) = 0 \\
 & y = \tan(10\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_4) = 0 \\
 & y = \tan(11\,\frac{2\pi}{12})\,(x - \tilde{\sigma}_5) = 0
\end{array}$$

where

$$\sigma_k = \sum_{j=0}^{k-1} \sin\left(\frac{7-2j}{9\pi}\right) \text{ and } \tilde{\sigma}_k = \sum_{j=0}^{k-1} \sin\left(\frac{2j+1}{12\pi}\right).$$

CHAPTER 2

Exercises

1.[L1] Interesting Surfaces

a) Find some interesting surfaces in either parametric form $\{x(s, t), y(s, t), z(s, t)\}$ or in the explicit form $z = z(x, y)$ by looking in the references cited in the sections of this chapter, and plot them using `ParametricPlot3D` or `Plot3D`.

b) Construct some interesting surfaces by using various pieces and gluing them together. Display them using `Show[Graphics3D[…]]` instead of using `ParametricPlot3D` or `Plot3D`.

2.[L2] Warped Torus, Twisted Torus, Massive Torus Wireframe, Interlocked Tori

a) Using `ParametricPlot3D`, plot a torus whose surface is warped periodically in two directions.

b) Using `ParametricPlot3D`, plot a twisted torus with a noncircular cross section.

c) Make a wireframe picture of a torus in which the wireframe consists of massive “beams”.

d) Make a picture of a torus having a rough surface.

e) Make a picture of a torus with the surface made from hexagons.

f) Make a picture of a torus with the surface made from random triangles.

g) Make a picture of a torus whose surface is made from interwoven tubes.

h) Generate a torus made from 4×4 points. Smooth the torus by repeatedly cutting off all vertices.

i) Construct a part of an array of interlocking tori periodic in three directions. Apply various “interesting” transformations to the array of tori.

j) Use parts of planes, circular cylinders, and tori to construct smooth double and triple tori (meaning that the surface normal is a continuous function everywhere on their surface). Color the various surface parts differently. Use a global coordinate transformation $\{\tilde{x}(x, y, z), \tilde{y}(x, y, z), \tilde{z}(x, y, z)\}$ to intermingle to two arms of the double torus.

3.[L1] Platonic Bodies with Holes, Edges of Platonic Bodies, Iteratively Reflected Dodecahedra

a) Implement a function `makeHoles` that shrinks the boundary faces of a convex polyhedron somewhat, and cuts a hole in each face. In addition, expand the remaining faces to 3D beams. Test your function on the Platonic solids.

b) Determine and illustrate all possibilities of a set of edges of a Platonic solid that is fixed in space, such that every vertex belongs to exactly one of the selected edges. When working with a fixed edge numbering, by taking into account constellations that can be transformed into each other by rotations and reflections.

c) Start with a regular dodecahedron and repeatedly reflect the dodecahedron on all of its faces outwards. Sort the resulting dodecahedra by their distance from the origin and display the groups of dodecahedra of equal distance.

4.[L1] Snail

Construct a shell for a snail. There are many possibilities; for a few suggestions, see pages 28–29 of the *Mathematica* V.2 manual [111✱]. Pay attention to the efficiency of your program.

5.[L2] Harlekin's Dodecahedron, Trinomial Theorem Visualization

a) Construct a dodecahedron whose faces are divided into 10 congruent triangles. Color these triangles black and white so that only triangles of different colors meet along the edges of the polygons.

b) Write a one-liner that visualizes the trinomial theorem

$$(x_1 + x_2 + \cdots + x_n)^3 = \sum_{k_1=0}^{3} \sum_{k_2=0}^{3} \cdots \sum_{k_n=0}^{3} \delta_{3,k_1+k_2+\cdots+k_n} (3;\, k_1,\, k_2,\, \ldots,\, k_n)\, x_1^{k_1}\, x_2^{k_2} \cdots x_n^{k_n}.$$

in the following way [72✱]: The left side can be interpreted as a cube with edge length $l = x_1 + x_2 + \ldots + x_n$. After summing all n^3 terms on the right-hand side, a term $p\, x_{j_1}\, x_{j_2}\, x_{j_3}$ can be interpreted as p cuboids with edge lengths x_{j_1}, x_{j_2}, and x_{j_3}. Subdivide the original cube with edge length l into the n^3 cuboids arising from the described expansion and color all cuboids with identical edge lengths the same way.

6.[L3] Blending Three Tori, Blending Two Tubes, Loop Subdivision, $\sqrt{3}$ Subdivision, Line Averages

a) Suppose three identical tori are positioned so that pairs of them touch. Blend them together with appropriate surfaces, so that an aesthetically pleasing picture results.

b) One possible way to construct a smooth transition between two intersecting surfaces is to roll a sphere along the intersection. Then the envelope of the sphere forms such a smooth transition (called constant radius blend) [67✱], [61✱], [81✱], [78✱], [66✱], [99✱], [82✱], [75✱], [100✱], [104✱]. Construct a constant radius blend for two perpendicular cylinders with circular cross sections. Show the resulting smooth transition for one cylinder along the z axis with radius 1 and one cylinder along the x axis with radius 1 (and in a second example, with radius 2).

c) Use the Loop subdivision scheme ([109✱], [63✱], [97✱], [83✱], [106✱], [89✱], [90✱], [103✱], [112✱], [71✱], [59✱], [96✱], [86✱], [73✱], [91✱], [113✱]), to make a smooth version of a solid cube wireframe. In the Loop subdivision scheme a triangle of a self-intersection-free, boundary-free surface is subdivided into four new ones in a Sierpinski triangle-like manner. The coordinates $V^{(l)}_{\text{vertex};\,k}$ of the triangle vertices are in the lth stage of the subdivision process given by

$$V^{(l)}_{\text{vertex};\,k} = \frac{1}{8}\left(\alpha_n\, V^{(l-1)}_{\text{vertex};\,k} + \beta_n \sum_{m=1}^{n} V^{(l-1)}_{\text{vertex};\,m}\right)$$

where the sum runs over all n neighboring vertices of the vertex $V^{(l-1)}_{\text{vertex};\,k}$. The constants α_n and β_n are given as:

$$\alpha_n = 8 - n\,\beta_n$$
$$\beta_n = \frac{1}{n}\left(5 - \frac{1}{8}\left(3 + 2\cos\!\left(\frac{2\pi}{n}\right)\right)^2\right).$$

The coordinates of the new edge vertex $V^{(l)}_{\text{edge};\,q}$ along the edge $\{V^{(l-1)}_{\text{vertex};\,k},\, V^{(l-1)}_{\text{vertex};\,m}\}$ are given by

$$V^{(l)}_{\text{edge};\,q} = \frac{3}{8}\,(V^{(l-1)}_{\text{vertex};\,k} + V^{(l-1)}_{\text{vertex};\,m}) + \frac{1}{8}\,(V^{(l-1)}_{\text{vertex};\,o} + V^{(l-1)}_{\text{vertex};\,p})$$

where $V_{\text{vertex};\,o}^{(l-1)}$, $V_{\text{vertex};\,p}^{(l-1)}$ are the coordinates of the third vertices of the two triangles that share the edge $\{V_{\text{vertex};\,k}^{(l-1)}, V_{\text{vertex};\,m}^{(l-1)}\}$.

Make an animation that shows a smooth transition from the thickened wireframe of a cube to the smoothed one.

d) Use the $\sqrt{3}$-subdivision scheme [84✱], [63✱], [85✱], [54✱], [94✱], [79✱], [113✱], to smooth a 3D sketch of a double torus. In the $\sqrt{3}$-subdivision scheme, the triangles of a self-intersection-free, boundary-free surface are subdivided in the following manner. In the lth stage of the subdivision process, two new triangles are formed each edge of the triangles of the $l-1$th stage by flipping the edge to connecting the centers of gravity of the two neighboring triangles. The endpoints of the flipped edge are updated according to

$$V_{\text{vertex};\,k}^{(l)} = (1-\alpha_n)\, V_{\text{vertex};\,k}^{(l-1)} + \frac{\alpha_n}{n} \sum_{m=1}^{n} V_{\text{vertex};\,m}^{(l-1)}$$

where the sum runs over all n neighboring vertices of the vertex $V_{\text{vertex};\,k}^{(l-1)}$. The constant α_n is given as $\alpha_n = (4 - 2\cos(2\pi/n))/9$. Carry out the $\sqrt{3}$-subdivision scheme on a stellated icosahedron.

e) Implement a function `averageLineGraphic3D`, that, given a closed line `Line[`*l*`]`, generates a graphic of the surface obtained by averaging successively increasing neighborhoods of all points of the line. Visualize the resulting surfaces for various initial lines.

7.$^{\text{L1}}$ Projective Plane, Genus *k* Surfaces

a) Display a model of a projective plane in $\mathbb{R}^3$. This is a generalization of the Klein bottle, and it is also self-intersecting. The projective plane arises by identifying opposite sides of a square with opposing orientations.

b) Given a $2k$-gon with clockwise oriented edges, there are $(2k-1)!!$ possibilities to pair the $2k$ edges. Gluing such pairs edges together (in the sense discussed in Subsection 2.3.5) results in closed (potentially intersecting) surfaces (so-called maps). In the gluing process originally different vertices coincide and v different vertices remain. The genus g of the resulting surface is (according to Euler's formula) $g = (k+1-v)/2$. For $2 \le 2k \le 12$, calculate the genus of all possible surfaces formed in the described way.

8.$^{\text{L2}}$ Pyramid

Construct a pyramid whose horizontal cross sections are best approximations of a circle on a square grid. It should consist of small cubes whose vertices all have integer coordinates, and whose vertical walls are parallel to the x,z-plane or the y,z-plane.

9.$^{\text{L3}}$ Fractal Mountains, Long Random Walk, Random Walk on a Sphere

a) Construct a fractal surface by iterative subdivision of a square and random selection of the height of the corresponding new points. Try to get an efficient implementation. Determine how to display these fractal surfaces most efficiently.

b) Find a self-intersection free random walk on a 3D cubic lattice that has at least 10000 steps.

c) Visualize a random walk on a sphere.

10.$^{\text{L1}}$ Glued Strip

Visualize the construction of a one-sided surface by appropriately gluing together edges of a cylindrical surface (without cutting).

11.[L1] Wrapping

a) Display a sphere with strips winding from the north pole to the south pole.

b) Wrap the surface of a torus with strips.

c) Make a picture of a torus. Then take this 2D picture of this torus and warp it around a 3D torus.

12.[L2] Projection onto Cube and Dodecahedron, Intersections of Planes

a) Project a curve lying on a sphere onto a cube containing the sphere (where the cube and sphere have the same center). This means that each point P on the sphere is mapped onto the point on the cube that lies on the ray through P and the center.

b) Project a curve in 3D onto a dodecahedron. This means that each point P of the curve is mapped onto the point on the dodecahedron that lies on the ray through P and its center.

c) Make an animation showing how the intersections of a set of randomly oriented planes (with time-dependent orientation) through the origin intersect with the unit sphere.

13.[L2] Alexander's Horned Sphere, Polyhedral Caustic

a) Visualize a so-called Alexander's horned sphere ([55✱], see also [98✱], and [101✱]). Such surfaces arise in topology and consist of a sphere with two attached horns. At the tips of these horns, two more horns intertwine; on their tips, two more horns intertwine, and so on.

b) The wavefront caustics of a surface $\{\xi, \eta, \zeta\}$ in 3D are the points $\{x, y, z\}$, where the map $\{x, y, z\} = \{\xi, \eta, \zeta\} + s\, n(\{\xi, \eta, \zeta\})$ becomes singular [58✱], [95✱]. Here $n(\{\xi, \eta, \zeta\})$ is the surface normal at the point $\{\xi, \eta, \zeta\}$ and we consider the map as a function of two the surface parametrizing variables and of s. For the surface implicitly described by $\sum_{k=1}^{8} |\{\xi, \eta, \zeta\} - v_k| = constant$, where the v_k are the vertices of a cube, parametrize the surface in spherical coordinates φ, ϑ and visualize its wavefront caustics [80✱].

14.[L2] Sliced Möbius Strip

Given a list of polygons (e.g., from `ParametricPlot3D`), find an associated list of lists of the polygons that are connected to each other (your function may not give real values for all parameters). Test the program on a Möbius strip that is repeatedly cut down the middle.

15.[L2] Perspective Drawing, Viewpoint in Absolute Coordinates, Hidden Edges

a) Consider a cubical wireframe (made out of lines). Create your own program to display the wireframe, and compare the resulting images with the perspective 3D views obtained using `Show[Graphics3D[`*rectangularWireFrame*`], `*options*`]` by displaying the two wireframes on top of each other (use the default values for the options `ViewPoint`, `ViewCenter`, and `ViewVertical`).

b) Extend `Plot3D` so that one can specify the viewpoint in the coordinate system of the surface displayed.

c) Given a `Graphics3D`-object, write a function that displays the graphic with the visible polygon edges as solid lines and the invisible polygon edges as dashed lines.

16.[L2] Platonic Solids on Platonic Solids

a) Write a program that places given Platonic solids at the vertices of other Platonic solids.

b) Write a program that generates random clusters of a given Platonic solid. The individual Platonic solids are glued together in a matching way at their faces [68✱]. (Take care that the bodies do not intersect.)

c) Generate and visualize a random cluster of Platonic solids that are glued together in each case along one of their edges. Let the number of the Platonic solids of each kind be determined by a given list of probabilities. (Take care that the bodies do not intersect.)

17.[L2] 120-Cell, Kochen–Specker Theorem

a) Make a wireframe picture of the cross section $w = 0$, $x = 0$, $y = 0$, and $z = 0$ of the so-called 120-cell [107✱]. The 120-cell is one of the regular polyhedra in four dimensions. The vertices of the 120-cell are given by ([76✱], [105✱], [62✱], [87✱], [69✱], and http://freeabel.geom.umn.edu/docs/forum/polytope/, and http://members.aol.com/Polycell/index.html)are all

$$\{\pm 2, \pm 2, 0, 0\}$$
$$\left\{\pm \sqrt{5}, \pm 1, \pm 1, \pm 1\right\}$$
$$\{\pm \phi, \pm \phi, \pm \phi, \pm \phi^{-2}\}$$
$$\{\pm \phi^2, \pm \phi^{-1}, \pm \phi^{-1}, \pm \phi^{-1}\}$$

and all even permutations of

$$\{\pm \phi^2, \pm \phi^{-2}, \pm 1, 0\}$$
$$\left\{\pm \sqrt{5}, \pm \phi^{-1}, \pm \phi, 0\right\}$$
$$\{\pm 2, \pm 1, \pm \phi, \pm \phi^{-1}\}$$

Calculate the 2D and 3D faces of the 120-cell.

After identifying antipodal vertices by forming rays in $\mathbb{R}^4$, is it possible to color these rays in so that there is exactly one green and three red ones in any 4-tuple of orthogonal rays [56✱], [57✱]?

18.[L3] Folding a Dodecahedron, Projected Dodecahedra, Random Polyhedra, Five Cubes in a Dodecahedron, Bands Around a Dodecahedron

a) Generate an animation showing how to fold a dodecahedron by starting the folding process from a planar arrangement of pentagons. [88✱]

b) Take a dodecahedron and project other randomly oriented dodecahedra on its faces (one per face). Now project such dodecahedra on the faces of another dodecahedron. Make an animation where all involved dodecahedra rotate around their centers in a random manner.

c) Make an animation of a "continuously changing" polyhedron (allow for truncated vertices).

d) It is possible to inscribe five cubes into a dodecahedron in such a way that the vertices of the cubes coincide with vertices of the dodecahedron [70✱], [65✱], [64✱]. Make a graphic that shows these five cubes inside a dodecahedron. Then make an animation that shows the cubes rotating along their main diagonal.

e) Calculate how to interweave 10 planar bands, such that the resulting arrangement has dodecahedral symmetry and each band can be slung around a dodecahedron in such a way that each band crosses the second nearest neighbor edges of the dodecahedron faces [77✱], [110✱].

19.^{L3} Knot with Knots, Lizards on Knot, Gear Chain

a) Cover the surface of a knot with smaller copies of the same knot in such a way that they are interlocked with their neighbors.

b) Put the lizard pattern from Subsection 1.5.8 on a knot so that the lizards "fit together" everywhere.

c) Make an animation of a chain of gears positioned along a trefoil knot that rotate.

20.^{L3} 3D Peano Polygon

For the L-system

```
l = LSystemWithFAndLAndR[
{"-", "L"},{"L" -> Characters["LF+RFR+FL-F-LFLFL-FRFR+"],
            "R" -> Characters["-LFLF+RFRFR+F+RF-LFL-FR"]}, 4, 2Pi/4]
```

from subsection 1.5.10. Display the polygon `Polygon[Join[{{-1, 0}}, l, {{-1, l[[-1, -1]]}}]]` embedded into 3D with finite thickness.

21.^{L2} Tetraview Animation, Kepler Equation Riemann Surface

a) The graph of a complex-valued function $w(z)$ can be considered as a hypersurface in $\mathbb{R}^4$ with the four real coordinates $\{\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w(z)), \mathrm{Im}(w(z))\}$ [92✱], [60✱]. Carrying out a rotation in 4D and projecting to the first three coordinates allows to continuously change from $\{\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w(z))\}$ to $\{\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Im}(w(z))\}$ (the so-called tetraview). Make an animation that smoothly connects all six possible these two and other projections for the Riemann surface of the function $w(z) = (1 - z^2)^{1/3}$.

b) Make an animation that shows how the Riemann surface $y(x)$ of the Kepler equation $x = y + \varepsilon \sin(y)$ [108✱], [93✱], [102✱], [74✱] changes as ε takes on different complex values. Use $\mathrm{Re}(y(x))$ as a faithful representation of the Riemann surface.

21.^{L3} "Sierpinski Plant"

Construct a randomized "Sierpinski plant". A Sierpinski plant is based on the recursive Sierpinski subdivision of a triangle. After each subdivision stage, let the new triangles "grow outwards". Choose the growth directions and growth lengths randomly.

CHAPTER 3

Exercises

1.[L1] Reducible Fractions

a) Visualize the irreducible fractions of the form p/q in the p,q-plane. Approximately what percentage of all possible fractions are irreducible?

b) The probability w that a fraction is reducible is given by

$$w = \frac{1}{4} + \sum_{j=2}^{\infty} \frac{1}{p_j^2} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2}\right).$$

(This formula follows easily by considering which numbers can be canceled from a fraction; see, e.g., [134✱]). Here, p_j is the jth prime number (in *Mathematica* this is by `Prime[j]`). Compute an approximate numerical value for the probability by chopping off the infinite sum after a finite number of terms (≈ 200). Implement an exact computation of w for a finite upper limit on the sum, which is at least three times as fast as the naive implementation `wNaiv[nMax]`.

```
wNaiv[nMax_] := Sum[1/Prime[j]^2 Product[1 - Prime[i]^-2, {i, j - 1}],
                    {j, 2, nMax}] + 1/4
```

c) Consider the square $n \times n$ matrix a_{ij} ($1 \leq i, j \leq 400$) with entries 1 if $\gcd(i, j) = 1$ and 0 else. Find all connected subparts of this matrix. (An element $a_{ij} = 1$ is connected to the $a_{i+1\,j}$, $a_{i-1\,j}$, $a_{i\,j+1}$, $a_{i\,j-1}$ that have the value 1.

2.[L1] Saddle Points

Examine the function $z(x, y)$ defined as

$$z(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - k_x\, x - k_y\, y - k_{xy}\, x\, y.$$

to find the positions of the local extrema and the saddle points as functions for various values of $k_x, k_y, k_{xy} \geq 0$.

3.[L1] Chladny Tone Figures, Tetrahedron Eigenfunctions

a) Chladny tone figures correspond to collections of powder/sand at the knot lines of an oscillating plate ([151✱], [144✱], [130✱], [171✱], [155✱], [163✱], [178✱], [154✱], [162✱], and [120✱] ; for the accumulation of powder at the antinodes, see [168✱]). The corresponding differential equation is the wave equation, $(\partial^2/\partial t^2 - \Delta)\, u(x, y, t) = 0$, where $u(x, y, t)$ is the vertical displacement of the plate. Assuming a harmonic time dependence, $u(x, y, t) = \cos(\omega\, t)\, v(x, y)$, the differential equation reduces to the Helmholtz equation $(\Delta - \omega^2)\, v(x, y) = 0$.

The spatial component $v(x, y)$ of the eigenfunctions for the square plate (the plate is $(0, \pi)\times(0, \pi)$) has the form:

$$v(x,y) = \sum_{n^2+m^2=const} c_{nm} \cos(n\,x)\cos(m\,y), \quad c_{nm} \text{ arbitrary}, n, m > 0, \text{ integers.}$$

Visualize a Chladny tone figure for a square plate for a sum with at least two terms.

b) The spatial component $v(x, y)$ of the eigenfunctions for a thin vibrating triangular plate with vertices $p_1 = \{0, 0\}$, $p_2 = \{\sqrt{3}/2, 0\}$ and $p_3 = \{\sqrt{3}/2, 1/2\}$

$$v(x,y) = \sum c_{nm}\Big(\sin\Big(\frac{2\pi}{3}\sqrt{3}\,(n-m)\,x\Big)\sin\Big(\frac{2\pi}{3}(3\,n-m)\,y\Big) +$$

$$(-1)^n \sin\Big(\frac{2\pi}{3}\sqrt{3}\,(2\,n-m)\,x\Big)\sin\Big(\frac{2\pi}{3}\,m\,y\Big) + (-1)^m \sin\Big(\frac{2\pi}{3}\sqrt{3}\,n\,x\Big)\sin\Big(\frac{2\pi}{3}(2\,m-3\,n)\,y\Big)\Big)$$

The c_{nm} arbitrary, $n \neq 0$, $m \neq 0$, $n \neq m$, $3\,n \neq m$, $3\,n \neq 2\,m$, $2\,n \neq m$, $n,m > 0$ integer.

(For a derivation, see [149✱], [153✱], [125✱], [123✱], [150✱], [148✱], [114✱], [143✱], [179✱], [164✱], [119✱], [145✱], [136✱], and [147✱]; for a generic right angle triangle, see [132✱]).

Visualize a Chladny tone figure for a triangular plate for some n,m. Take proper account of the geometry of the plate in your plot. (For the much more complicated analysis of the vibrations of real plates, see [133✱] and [131✱].)

c) The eigenfunctions $\psi_{l,m,n}(x, y, z)$ of the Helmholtz equation in a tetrahedron with vertices $\{-\pi/2^{1/2}, \pi/2^{1/2}, \pi/2\}$, $\{\pi/2^{1/2}, -\pi/2^{1/2}, \pi/2\}$, $\{\pi/2^{1/2}, \pi/2^{1/2}, \pi/2\}$, and $\{\pi/2^{1/2}, \pi/2^{1/2}, \pi/2\}$ and Dirichlet boundary conditions can be expressed as the following determinant [142✱], [169✱], [165✱]

$$\psi_{l,m,n}(x,y,z) = \left\| \begin{pmatrix} 1 & 1 & 1 & 1 \\ g_l(x,z,3) & g_l(y,-z,1) & g_l(-x,z,-1) & g_l(-y,-z,-3) \\ g_m(x,z,3) & g_m(y,-z,1) & g_m(-x,z,-1) & g_m(-y,-z,-3) \\ g_n(x,z,3) & g_n(y,-z,1) & g_n(-x,z,-1) & g_n(-y,-z,-3) \end{pmatrix} \right\|$$

where l, m, and n are pairwise different nonnegative integers and $g_\alpha(\xi, \eta, j) = \exp(i\,\alpha\,(\xi/2^{1/2} + \eta/2 + j\,\pi/4))$.

Calculate the eigenvalue of $\varepsilon_{l,m,n}$ of $-\Delta\,\psi_{l,m,n}(x, y, z) = \varepsilon_{l,m,n}\,\psi_{l,m,n}(x, y, z)$. Show that the $\psi_{l,m,n}(x, y, z)$ vanish at the faces of the tetrahedron, and show explicitly the orthogonality of the two eigenfunctions $\psi_{1,2,3}(x, y, z)$ and $\psi_{4,5,6}(x, y, z)$ in

$$\iiint_{\text{tetrahedron}} \psi_{1,2,3}(x,y,z)\,\overline{\psi_{4,5,6}(x,y,z)}\,dxdydz = 0.$$

Finally, show the interior nodal surface (not including the facial ones) of some of the $\psi_{l,m,n}(x, y, z)$.

4.^{L2} Liénard–Wiechert Potential

Examine the radiation due to a point charge moving on a circular path. For convenience, restrict yourself to the scalar potential $\varphi(\mathbf{r}, t)$ (the associated vector potential $\mathbf{A}(\mathbf{r}, t)$ is easily obtained from $\varphi(\mathbf{r}, t)$ as $\mathbf{A}(\mathbf{r}, t) = \mathbf{v}(t_{red})\,\varphi(\mathbf{r}, t)$). Here, $\varphi(\mathbf{r}, t)$ is given by the Liénard–Wiechert potential:

$$\varphi(\mathbf{r},t) = \frac{q}{c\,|\mathbf{r}-\mathbf{r}(t_{red})| - (\mathbf{r}-\mathbf{r}(t_{red})).\mathbf{v}(t_{red})}$$

where q is the electric charge, c is the speed of light, $\mathbf{r}$ is the observation point, t is the observation time, $\mathbf{r}(t)$ is the parametric representation of the path, t_{red} is the retarded time, that is, the time for which

$$t - t_{red} = \frac{|\mathbf{r} - \mathbf{r}(t_{red})|}{c}$$

and $\mathbf{v}(t_{red})$ is the speed of the charge at the retarded time. Solve the last equation iteratively; this is possible for a wide range of parameters. (For more on Lienárd-Wiechert potentials, see, e.g., [121✱], [157✱], [138✱], [137✱], [124✱], [128✱], [160✱], [156✱], [135✱], [172✱], [173✱], [174✱], [175✱], [176✱], [152✱], and [177✱].)

5.^{L2} Shallit–Stolfi–Barbé Plots, Mirrored Matrix

a) Construct Shallit–Stolfi–Barbé plots ([158✱], [116✱], and [117✱]). These are `ListDensityPlot`s of (possibly large) 2D matrices. The interesting thing here is the way in which the matrices are created. Given a function f of two variables, we associate a “big” function **F** with matrices as arguments. It is defined as follows. Let **A** and **B** be two matrices with elements a_{ij} $(1 \le i \le r_{\mathbf{A}},\ 1 \le i \le s_{\mathbf{A}})$ and b_{ij} $(1 \le i \le r_{\mathbf{B}},\ 1 \le i \le s_{\mathbf{B}})$, respectively. Then, $\mathbf{F}(\mathbf{A}, \mathbf{B})$ is an $r_{\mathbf{A}}\, r_{\mathbf{B}} \times s_{\mathbf{A}}\, s_{\mathbf{B}}$ matrix whose elements consist of outer products, which we compute by replacing each a_{ij} by the block matrix $f(a_{ij}, b_{kl})$, for $k = 1, \ldots, r_B$ and $l = 1, \ldots, s_B$.

Here is an example:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then, $\mathbf{F}(\mathbf{A}, \mathbf{B})$ is given by (this is basically the elementwise application of **F** to the Kronecker product)

$$\begin{pmatrix} f(a_{11}, b_{11}) & f(a_{11}, b_{12}) & f(a_{12}, b_{11}) & f(a_{12}, b_{12}) \\ f(a_{11}, b_{21}) & f(a_{11}, b_{22}) & f(a_{12}, b_{21}) & f(a_{12}, b_{22}) \\ f(a_{21}, b_{11}) & f(a_{21}, b_{12}) & f(a_{22}, b_{11}) & f(a_{22}, b_{12}) \\ f(a_{21}, b_{21}) & f(a_{21}, b_{22}) & f(a_{22}, b_{21}) & f(a_{22}, b_{22}) \end{pmatrix}.$$

Try to make the computation of the big matrix efficient. Iterate the **F** operation, and interpret the elements of the resulting matrix as gray levels for some functions **F** of your own.

b) Given a quadratic matrix, construct a density plot of multiple reflection of a matrix on all its sides. The new values arise from addition of all the mirrored ones.

6.^{L1} Alternatives to `DensityPlot`, Contour Plot of Functions with Poles

a) For an arbitrary function of two variables, compare the following three gray level plots:
- The plot produced by `DensityPlot`.
- The gray levels of the squares, and color them.
- The `Plot3D` with function values depending on the gray levels, and look at it from “high above”.

b) The eigenvalue equation for the Schrödinger equation $\psi_\varepsilon''(z; g, \lambda) + 2\, g\, \delta(z - \lambda)\, \psi_\varepsilon(z; g, \lambda) = \varepsilon(g, \lambda)\, \psi_\varepsilon(z; g, \lambda)$ with boundary conditions $\psi_\varepsilon(\pm 1; g, \lambda) = 0$ reads $\varepsilon(g, \lambda)^{1/2}\, (\cot(\varepsilon(g, \lambda)^{1/2}\, (1 - \lambda)) + \cot(\varepsilon(g, \lambda)^{1/2}\, (\lambda + 1))) = 2\, g$ [170✱], [126✱], [141✱], [129✱]. A direct attempt to visualize the spectrum $\varepsilon(g^*, \lambda)$ with

```
evEqs[ε_, g_, λ_] := Sqrt[ε] (Cot[Sqrt[ε] (1 + λ)] +
                              Cot[Sqrt[ε] (1 - λ)]) + 2g
ContourPlot[Evaluate[evEqs[ε, 10, λ]], {λ, -1, 1}, {ε, 0, 100},
            PlotPoints -> 120, Contours -> {0}, ContourShading -> False];
```

gives a very poor result. How can this be remedied?

7.[L1] Torus Equation

Starting with the formula $\sum_{i,j,k=0}^{n} a_{ijk}\, x^i\, y^j\, z^k = 0$, analyze several cross sections to find the implicit formula for a torus. Visualize the torus using `ContourPlot3D`.

8.[L2] Fractals

Fractal images appear everywhere these days, so we have to generate at least one. Look in the references for an image, and program it in *Mathematica* (or make up one of your own). Typically, it is necessary to do a lot of numerical calculation, so pay special attention to efficiency.

9.[L2] `ContourPlot3D` and O_h, Y_h-Symmetry

a) Plot the following surfaces with the help of the command `ContourPlot3D`.

$$\prod_{i=1}^{27} |(x, y, z) - (x_i, y_i, z_i)|^2 = 10^{11}$$

(here, the product encompasses all points $\{x_i, y_i, z_i\}$ generated by `Flatten[Outer[List, {0, -1, 1}, {0, -1, 1}, {0, -1, 1}], 2]`) and

$$(8\,x^4 - 8\,x^2 + 1) + (8\,y^4 - 8\,y^2 + 1) + (8\,z^4 - 8\,z^2 + 1) = 0 \text{ (after Cmutov [115 ✱]).}$$

Take advantage of the symmetry of the resulting surfaces to construct a plot using `ContourPlot3D[…]`.

b) Plot the following surface with the help of the command `ContourPlot3D`.

$$\prod_{i=1}^{30} ellipsoid_i = c$$

Here, the product encompasses all ellipsoids of rotation along with the edges of a dodecahedron. Choose the parameters (half axes of the 30 ellipsoids and c) so that the ellipsoids blend together at the vertices of the dodecahedron. Again, take advantage of the symmetry of the resulting surfaces to construct a plot using `ContourPlot3D[…]`.

Make contour plots of "random" surfaces having dodecahedral symmetry.

10.[L2] Icosahedron Equation, Belyi Functions

a) Make a `ContourPlot` of the real and imaginary parts of the left-hand side of Klein's icosahedral equation in dependence of the complex variable z. The icosahedral equation is

$$((z^{20} + 1) - 228\,(z^{15} - z^5) + 494\,z^{10})^3 + 1728\,z^5\,u(z^{10} + 11\,z^5 - 1)^5 = 0.$$

(The name icosahedral equation comes from the fact that the "interesting points" in the `ContourPlot` are just the stereographic projections of the midpoints of the faces (for $u = 0$), the midpoints of the edges (for $u = 1$), and the corners ($u = \infty$), see [140✱], [161✱], and [146✱]). Use symmetry of the function to reduce the region where actual calculations are carried out.

b) Generate a graphic that shows for which complex z the polynomial $p(z)$ (defined below [146✱], [159✱]) takes on positive

$$p(z) = p_T(p_D(p_I(z))) - 1;$$
$$p_T(z) = -\frac{27\,z^2}{(z-4)^3}$$
$$p_D(z) = 4\,z\,(1-z)$$
$$p_I(z) = \frac{(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1)^3}{1728\;z^5\,(z^{10} - 11\,z^5 - 1)^5}$$

11.^{L2} Zeros of Polynomial on the Riemann Sphere, Branch Cuts of Hyperelliptic Curve

a) Take a random polynomial $poly(z)$ (say of order around 20) and make a picture of the line $\mathrm{Re}(poly(z)) = 0$ and $\mathrm{Im}(poly(z)) = 0$ on the Riemann sphere.

b) Generate an animation of the branch cuts of a random hyperelliptic curve of the form

$$w(z) = \sqrt{\prod_{k=0}^{n} (z - (z_k^{(0)} + r_k\, e^{i\varphi}))}$$

as a function of φ. Here $z_k^{(0)}$ are random complex numbers and r_k are random real numbers.

12.^{L2} Charged *Mathematica*, Radial-Azimuthal Animation

a) Make a contour plot of the equipotential lines of the potential for the charged letters (modeled for simplicity by lines in a plane) of *Mathematica*.

b) Make a “random” polygon (potentially having holes) with more than 100 vertices. Then charge the edges of this polygon and visualize the equipotential curves in the inside of the polygon.

c) Predict the qualitative behavior of the following animation. Why are the pure functions around `Compile` and `ContourPlot` needed?

```
radialAzimuthalTransition[{o_, k_, ν_}, l_, pp_, frames_, opts___] :=
Module[{t, x, y, f, λ = 1.1 l, border},
f = Compile[#1,
     Module[{r = Sqrt[x^2 + y^2], φ = ArcTan[x, y]}, #2]]&[{t, x, y},
     Exp[2 I t] Sum[Exp[I 2Pi Random[]]*
     (Cos[t] + Sin[t] Exp[I Random[] k r])*
     (Cos[t] Exp[I (Random[Integer, ν {-1, 1}] φ +
                    2 Pi Random[])] + Sin[t]), {o}]];
border = {{GrayLevel[1], Polygon[{{l, 0}, {λ, λ}, {-λ, λ},
                                  {-λ, -λ}, {λ, -λ}, {l, 0},
                                   Sequence @@ #}]},
          {GrayLevel[0], Thickness[0.01], Line[#]}}&[
           Table[l{Cos[-φ], Sin[-φ]}, {φ, 0, 2Pi, 2Pi/60}]] // N;
Do[Function[t,
   ContourPlot[Im[f[t, ξ, η]], {ξ, -l, l}, {η, -l, l},
               opts, Compiled -> False, PlotPoints -> pp,
               PlotRange -> All, Contours -> {0}, Frame -> False,
               ColorFunctionScaling -> False, Epilog -> border,
               ColorFunction -> (If[# < 0, Hue[t/Pi],
```

```
                                         Hue[t/Pi + 1/2]]&)]][τ],
  {τ, 0, Pi/2 (1 - 1/frames), Pi/2/frames}]]
```

```
radialAzimuthalTransition[{20, 20, 30}, 5, 100, 12]
```

Make a similar, contour plot-based animation showing the transition from a set of flower-like contour curves into concentric contours.

d) Make a contour plot-based animation of the sum of the inverse distances of a set of randomly moving, nonintersecting disks.

13.^{L3} `Plot3D` with Contour Lines, 3D Contour Plot

a) `Plot3D` does not have an option to show the contour lines of a surface. Extend `Plot3D` in such a manner that contour lines are shown on the surface. Try to avoid the use of `ContourPlot`.

b) Implement a function `ContouredPlot` that takes a `SurfaceGraphics`-object and colors the parts of the surface that lie within specified contour (z-) values. Show various examples of the application of `ContouredPlot`.

14.^{L1} Singular Points

Calculate nonvanishing polynomials $p(x, y, z)$ that have the following properties:

■ $p(0, 0, 0) = 0$

■ all derivatives up to order o vanish at the origin.

For instance, for $o = 2$, these conditions mean

$$\frac{\partial p(x,y,z)}{\partial x} = \frac{\partial p(x,y,z)}{\partial y} = \frac{\partial p(x,y,z)}{\partial z} = 0$$

$$\frac{\partial^2 p(x,y,z)}{\partial x^2} = \frac{\partial^2 p(x,y,z)}{\partial y^2} = \frac{\partial^2 p(x,y,z)}{\partial z^2} = \frac{\partial^2 p(x,y,z)}{\partial x\,\partial y} = \frac{\partial^2 p(x,y,z)}{\partial x\,\partial z} = \frac{\partial^2 p(x,y,z)}{\partial y\,\partial z} = 0.$$

For $o = 2$, $o = 3$, $o = 4$, and $o = 5$, visualize how the surface $p(x, y, z)$ looks near the origin.

15.^{L3} Gauss–Bonnet Theorem, Interlocked Double and Triple Tori

a) Along the lines of Section 3.3, construct a flat double and flat triple torus. Color each torus according to its Gauss curvature. For a surface of the form $z = z(x, y)$, the Gauss curvature is given by [127✱], [122✱]

$$K = \frac{\frac{\partial^2 z(x,y)}{\partial x^2}\frac{\partial^2 z(x,y)}{\partial y^2} - \frac{\partial^2 z(x,y)}{\partial x\,\partial y}}{\left(1 + \left(\frac{\partial z(x,y)}{\partial x}\right)^2 + \left(\frac{\partial z(x,y)}{\partial y}\right)^2\right)^2}.$$

Use the polygons that form the double and triple torus to approximately numerically check that they are really double and triple tori by making use of the Gauss–Bonnet formula [118✱], [127✱]:

$$\frac{1}{4\pi}\oiint_{g\text{–torus}} K\, dA = 1 - g$$

Here, g is the number of holes of the surface and the integration extends over the whole surface of the g-torus.

b) Take some copies of a flat double torus, and some copies of a flat triple torus as well some simple tori, and interlink them in such a way that through each of the double and triple tori holes exactly one of the other arms goes, and none of the double and triple tori holes stays unfilled. Visualize at least ten topologically different situations.

16.[L2] Circular Contour Plots, Isophotes on Supersphere

a) The inverse elliptic nome function $q^{-1}(z)$ (in *Mathematica* `InverseEllipticNomeQ`) is the inverse of the elliptic nome function $q(z)$. It is only defined inside the unit circle, and the unit circle is the natural boundary of analyticity. Construct contour plots of the real and imaginary parts of $q^{-1}(z)$.

b) Make an animation showing how the isophotes on a supersphere $x^4 + y^4 + z^4 = 1$ change as the eyepoint changes. (The isophotes [167✱], [166✱], [139✱] of a surface $\mathcal{S}$ and an eyepoint $\mathbf{P}$ are the curves $\mathbf{n}(\mathbf{x}).\mathbf{e}(\mathbf{x}) = constant$ on $\mathcal{S}$. Here $\mathbf{n}(\mathbf{x})$ is the normal vector and $\mathbf{e}(\mathbf{x})$ is the unit vector from $\mathbf{x}$ on $\mathcal{S}$ to $\mathbf{P}$. Display the isophotes as sharp lines.

17.[L3] Structured Knot

Take an implicitly defined, "interesting"- looking surface, periodic in one direction, and bend the surface along a knot.

18.[L3] Finite Thickness Contour Surfaces

a) Implement an option setting `IsoContourSurfaceThickness[`δ`]` for the option `ContourStyle` of the function `ContourPlot3D` that renders the contour surfaces as "sheets" of a thickness δ.

b) Given an isocontour surface (say as a result of using the function `ListContourPlot3D`), make a sheet of finite thickness out of the contour surface. Do not assume that the contour surface is the result of an explicitly known function $f(x, y, z)$.

19.[L2] *n*-gons in `ContourPlot3D`

Make a 3D contour plot of a random function (like `f[x_, y_, z_] := Random[Real, {-1, 1}]`). Determine the frequency of n-gons ($n = 3, 4, 5, 6, 7$) in the resulting graphic. Compare the frequencies with the theoretical probabilities.

20.[L2] Double Torus with Texture

Cover the surfaces of a double torus with a dense Truchet-like pattern (this means with a nonintersecting, interlocking set of curves).

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