Sample Section of
The Mathematica GuideBook for Symbolics

1.10 Three Applications

■ 1.10.0 Remarks

In this section, we will discuss three larger calculations. Here, "larger" mainly refers to the necessary amount of operations to calculate the result and not so much to the number of lines of *Mathematica* programs to carry it out. The first two are "classical" problems. Historically, the first one was solved in an ingenious method. Here we will implement a straightforward calculation. Carrying out the calculation of an extension of the second one ($\cos(2\pi/65537)$) took more than 10 years at the end of the nineteenth century. The third problem is a natural continuation from the visualizations discussed in Section 3.3 of the Graphics volume [1751✶]. The code is adapted to *Mathematica* Version 5.1. As mentioned in the Introduction, later versions of *Mathematica* may allow for a shorter implementation and more efficient implementation.

■ 1.10.1 Area of a Random Triangle in a Square

In the middle of the last century, J. J. Sylvester proposed calculating the expectation value of the convex hull of n randomly chosen points in a plane square. For $n = 1$, the problem is trivial, and for $n = 2$, the question is relatively easy to answer. For $n \geq 3$, the straightforward formulation of the problem turns out to be technically quite difficult because of the multiple integrals to be evaluated. In 1885, M. W. Crofton came up with an ingenious trick to solve special cases of this problem. (His formulae are today called Crofton's theorem.) At the same time, he remarked:

The intricacy and difficulty to be encountered in dealing with such multiple integrals and their limits is so great that little success could be expected in attacking such questions directly by this method [direct integration]; and most of what has been done in the matter consists in turning the difficulty by various considerations, and arriving at the result by evading or simplifying the integration. [1041✶]

The general setting of the problem is to calculate the expectation value of the $\min(n-1, d)$-dimensional volume of the convex hull of n points in d dimensions, for instance, the volume of a random tetrahedron formed by four randomly chosen points in $\mathbb{R}^3$. For details about what is known, the Crofton theorem and related matters, see [35✶], [264✶], [567✶], [265✶], [1228✶], [840✶], [1041✶], [1249✶], [279✶], and [1421✶]. For an ingenious elementary derivation for the $n = 3$ case, see [1608✶]; for a tetrahedron in a cube, see [1926✶]. For the case of a tetrahedron inside a tetrahedron, see [1207✶].)

In this subsection, we will show that using the integration capabilities of *Mathematica* it is possible to tackle such problems directly—this means by carrying out the integrations. (This subsection is based on [1748✶].)

In the following, let the plane polygon be a unit square. We will calculate the expectation value of the area of a random triangle within this unit square (by an affine coordinate transformation, the problem in an arbitrary convex quadrilateral can be reduced to this case).

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Here is a sketch of the situation.

```
In[1]:= With[{P1 = {0.2, 0.3}, P2 = {0.8, 0.2}, P3 = {0.4, 0.78}},
    Show[Graphics[
            {{Thickness[0.01],
              Line[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}]},
             {Thickness[0.002], Hue[0], Line[{P1, P2, P3, P1}]},
             {Text["P1", {0.16, 0.26}], Text["P2", {0.84, 0.16}],
              Text["P3", {0.40, 0.82}]}}],
         AspectRatio -> Automatic]]
```
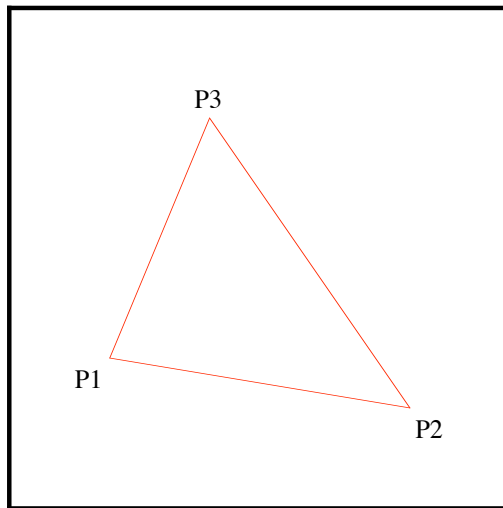


```
Out[1]=  - Graphics -
```

Let $\{x_1, y_1\}$, $\{x_2, y_2\}$, and $\{x_3, y_3\}$ be the vertices of the triangle. Then, the area A_T of the triangle is given by the absolute value of the determinant:

$$A_T = \frac{1}{2} \left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\| = \frac{1}{2} |x_3\, y_1 - x_2\, y_1 + x_1\, y_2 - x_3\, y_2 - x_1\, y_3 + x_2\, y_3|.$$

This means the integral to be calculated is

$$A = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 |x_3\, y_1 - x_2\, y_1 + x_1\, y_2 - x_3\, y_2 + x_2\, y_3 - x_1\, y_3|\, dy_3\; dx_3\; dy_2\; dx_2\; dy_1\; dx_1 .$$

Depending on the orientation of the triangle, the determinant changes sign, and there are two obvious possibilities to take care of this. The first would be to use $((x_3\, y_1 + x_1\, y_2 - x_3\, y_2 - x_1\, y_3 + x_2\, y_3 - x_2\, y_1)^2)^{1/2}$ as the integrand, and the second would be to divide the six-dimensional integration region into such subregions, so that inside each the expression $x_3\, y_1 + x_1\, y_2 - x_3\, y_2 - x_1\, y_3 + x_2\, y_3 - x_2\, y_1$ does not change sign.

We will follow this second approach here, because it makes interesting use of various *Mathematica* functions.

A subdivision of the six-dimensional unit cube into regions where the expression $x_3\, y_1 - x_2\, y_1 + \cdots$ has a constant sign can be most easily obtained using the cylindrical algebraic decomposition. Because for integration purposes we are only interested in

$$x_3 \, y_1 - x_2 \, y_1 + \cdots$$

subregions that have a nonvanishing six-dimensional Lebesgue measure, we use the function `GenericCylindricalAlge`‑ `braicDecomposition`. Here is the cylindrical algebraic description of the areas where the area of the triangle is positive; it can be calculated quickly.

In[2]:= `AppendTo[$ContextPath, "Experimental`"];`

In[3]:= `signedTriangleArea = -x2 y1 + x3 y1 + x1 y2 - x3 y2 - x1 y3 + x2 y3 > 0;`
`unitCube6D = 0 < x1 < 1 && 0 < y1 < 1 && 0 < x2 < 1 && 0 < y2 < 1 &&`
`            0 < x3 < 1 && 0 < y3 < 1`

Out[4]= $0 < x1 < 1 \;\&\&\; 0 < y1 < 1 \;\&\&\; 0 < x2 < 1 \;\&\&\; 0 < y2 < 1 \;\&\&\; 0 < x3 < 1 \;\&\&\; 0 < y3 < 1$

In[5]:= `Timing[cad = GenericCylindricalAlgebraicDecomposition[`
`            (* the triangle area *) signedTriangleArea &&`
`            (* the six-dimensional unit cube *) unitCube6D,`
`                {x1, y1, x2, y2, x3, y3}];]`

Out[5]= $\{0.511 \text{ Second}, \text{Null}\}$

We convert the resulting nested expression to a list of single six-dimensional regions.

In[6]:= `l1 = cad[[1]] //. a_ && (b_ || c_) :> a && b || a && c;`

This is one of the resulting regions.

In[7]:= `l1[[99]]`

Out[7]= $0 < x1 < \dfrac{1}{2} \;\&\&\; 1 - x1 < y1 < 1 \;\&\&\; x1 < x2 < \dfrac{x1}{y1} \;\&\&\; \dfrac{-y1 + x2\,y1}{-1 + x1} < y2 < \dfrac{x1 - x2 + x2\,y1}{x1} \;\&\&$

$\dfrac{x1 - x2 + x2\,y1 - x1\,y2}{y1 - y2} < x3 < 1 \;\&\&\; \dfrac{x2\,y1 - x3\,y1 - x1\,y2 + x3\,y2}{-x1 + x2} < y3 < 1$

Because we want to integrate the area function over the regions, we will convert from $\xi_l < \xi < \xi_u$ to $\{\xi_l, \xi, \xi_u\}$, a form that can be easily used inside `Integrate` as the integration regions.

In[8]:= `regions = Apply[List, Apply[{#3, #1, #5} &, l1, {2}], {0, 2}];`

We have 216 individual regions.

In[9]:= `Length[regions]`

Out[9]= 216

There is no need to calculate the regions where the area of the triangle has a negative sign. The resulting integrals will give exactly the same contribution by symmetry of the problem. Here is the resulting description of one of the regions.

In[10]:= `regions[[111]]`

Out[10]= $\left\{\left\{x1, \dfrac{1}{2}, 1\right\}, \{y1, 0, 1 - x1\},\right.$

$\left\{x2, 0, \dfrac{-x1 + y1}{-1 + y1}\right\}, \left\{y2, \dfrac{-y1 + x2\,y1}{-1 + x1}, \dfrac{x1 - x2 + x2\,y1}{x1}\right\},$

$\left.\left\{x3, 0, \dfrac{x2\,y1 - x1\,y2}{y1 - y2}\right\}, \left\{y3, 0, \dfrac{x2\,y1 - x3\,y1 - x1\,y2 + x3\,y2}{-x1 + x2}\right\}\right\}$

Now, inside each of the 216 six-dimensional regions, we have to integrate the area as a function of $x_1, y_1, x_2, y_2, x_3, y_3$. The area is only a multilinear polynomial in each of the six integration variables. Because the limits of the integrations are rational functions of the outer integration variables, the integrals we deal with will be typically quite complicated and contain rational functions, logarithms, and polylogarithms in the result after carrying out some integrations.

We will not use *Mathematica*'s definite integration here. As mentioned the following in Subsection 1.6.2: If a definite integral contains parameters, *Mathematica* tries to determine if the integral exists for all parameters and if the integrand contains singular points. Because the integrals involved possess up to five parameters (the outer integration variables), this turns out to be quite a complicated integration. *Mathematica* can carry out the integrations, but some for some regions, it will be a time-consuming task. Here is a typical result for one of the regions. The result contains the transcendental numbers π and $\ln(2)$.

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```
In[11]:= area = Det[{{x1, y1, 1}, {x2, y2, 1}, {x3, y3, 1}}]/2;
```

```
In[12]:= Integrate[area, Sequence @@ regions[[2]]]
```

$$Out[12]= \frac{1}{288} \, (19 - 5\,\pi^2 + 23\,\text{Log}[2] + 30\,\text{Log}[2]^2)$$

On the other hand, by the construction of the integration regions and by the nature of the area function, we do not have to worry about any of these problems—inside each of the integration regions the integrand is a smooth, not sign-changing function. So, we use the much faster indefinite integration capabilities of *Mathematica* and calculate the definite integrals from the indefinite ones. If we are lucky, we could just substitute the integration limits into the indefinite integral to obtain the definite integral. But sometimes this might give indeterminate expressions. In such a situation, we must be more careful and must calculate the needed limits by using a series expansion around the integration endpoints.

Now, let us implement the integration described above: The function `multiDimensionalIntegrate` is our own multidimensional, definite integration function. It recursively calls the function `fastIntegrate`, defined below.

```
In[13]:= multiDimensionalIntegrate[int_, iters__, iter_] :=
          multiDimensionalIntegrate[fastIntegrate[int, iter], iters];

         multiDimensionalIntegrate[int_, iter_] := fastIntegrate[int, iter]
```

The function `fastIntegrate` is our 1D integrator. It carries out the indefinite integral first. We use `Limit` to determine the values of the integral at the integration limits. This will take care of potentially indeterminate expressions that would arise from blind substitution of the limit points into the indefinite integral.

```
In[15]:= fastIntegrate[integrand_, {x_, l_, u_}] :=
         Module[{indefiniteIntegral, lValue, uValue},
             (* the indefinite integral *)
             indefiniteIntegral = Integrate[LogExpand[integrand //. x -> ξ], ξ];
             (* the lower limit *)
             lValue = Limit[indefiniteIntegral, ξ -> l, Direction -> -1];
             (* the upper limit *)
             uValue = Limit[indefiniteIntegral, ξ -> u, Direction -> +1];
             Factor[Together[uValue - lValue]]]
```

To speed up the indefinite integration and the calculation of the limits, we apply some transformation rules implemented in `LogExpand` to the expressions. `LogExpand` splits all `Log[`*expr*`]` into as many subparts as possible to simplify the integrands. Because we know that the integrals we are dealing with are real quantities, we do not have to worry about branch cut problems associated with the logarithm function, and so drop all imaginary parts at the end.

```
In[16]:= LogExpand[expr_] := PowerExpand //@ Together //@ expr
```

Now, we have all functions together and can actually carry out the integration. To get an idea about the form of the expressions appearing in the six integrations, let us have a look at the individual integration results of the first region. (The indefinite integrals are typically quite a bit larger than the definite ones, as shown in the following results.)

This is the description of the first six-dimensional region.

In[17]:= **regions[[1]]**

Out[17]= $\left\{\left\{x1,\ 0,\ \frac{1}{2}\right\},\ \{y1,\ 0,\ x1\},\ \left\{x2,\ 0,\ \frac{-x1+y1}{-1+y1}\right\},\ \left\{y2,\ 0,\ \frac{x2\ y1}{x1}\right\},\right.$

$\left.\left\{x3,\ \frac{x2\ y1-x1\ y2}{y1-y2},\ 1\right\},\ \left\{y3,\ 0,\ \frac{x2\ y1-x3\ y1-x1\ y2+x3\ y2}{-x1+x2}\right\}\right\}$

After the integration with respect to y3, we get a rational function in the remaining five variables x3, y2, x2, y1, and x1.

In[18]:= **fastIntegrate[area, regions[[1, -1]]]**

Out[18]= $\dfrac{(-x2\ y1+x3\ y1+x1\ y2-x3\ y2)^2}{4\ (x1-x2)}$

After the second integration with respect to x3, we still have a rational function in five variables y2, x2, y1, and x1.

In[19]:= **fastIntegrate[%, regions[[1, -2]]]**

Out[19]= $\dfrac{(y1-x2\ y1-y2+x1\ y2)^3}{12\ (x1-x2)\ (y1-y2)}$

After the third integration with respect to y2, logarithmic terms emerge.

In[20]:= **fastIntegrate[%, regions[[1, -3]]]**

Out[20]= $\dfrac{1}{72\ x1^3\ (x1-x2)}$

$\left(y1^3\ \left(6\ x1^2\ x2-6\ x1^5\ x2-6\ x1\ x2^2-9\ x1^2\ x2^2+15\ x1^4\ x2^2+2\ x2^3+3\ x1\ x2^3+6\ x1^2\right.\right.$

$x2^3-11\ x1^3\ x2^3+6\ x1^6\ \text{Log}[-y1]-18\ x1^5\ x2\ \text{Log}[-y1]+18\ x1^4\ x2^2\ \text{Log}[-y1]-$

$6\ x1^3\ x2^3\ \text{Log}[-y1]-6\ x1^6\ \text{Log}\left[\left(-1+\frac{x2}{x1}\right)\ y1\right]+18\ x1^5\ x2\ \text{Log}\left[\left(-1+\frac{x2}{x1}\right)\ y1\right]-$

$\left.\left.18\ x1^4\ x2^2\ \text{Log}\left[\left(-1+\frac{x2}{x1}\right)\ y1\right]+6\ x1^3\ x2^3\ \text{Log}\left[\left(-1+\frac{x2}{x1}\right)\ y1\right]\right)\right)$

After the fourth integration with respect to x2, we still have a mixture of rational and logarithmic terms. The size of the result increases.

In[21]:= **fastIntegrate[%, regions[[1, -4]]] // Simplify**

Out[21]= $-\dfrac{1}{216\ x1^3\ (-1+y1)^3}$

$\Big(y1^3\ \big(-2\ x1^3 + 24\ x1^4 - 33\ x1^5 + 13\ x1^6 - 21\ x1^3\ y1 + 9\ x1^4\ y1 + 18\ x1^5\ y1 - 12\ x1^6\ y1 +$

$21\ x1^3\ y1^2 - 9\ x1^4\ y1^2 - 18\ x1^5\ y1^2 + 12\ x1^6\ y1^2 + 2\ y1^3 - 3\ x1\ y1^3 + 3\ x1^2\ y1^3 -$

$22\ x1^3\ y1^3 + 30\ x1^4\ y1^3 - 12\ x1^5\ y1^3 + 6\ x1^3\ (-1 + 3\ x1 - 3\ x1^2 + 2\ x1^3)\ (-1+y1)^3$

$\operatorname{Log}[-x1] + 6\ x1^3\ (-y1^3 + 3\ x1\ y1^3 - 3\ x1^2\ y1^3 + x1^3\ (1 - 3\ y1 + 3\ y1^2))\big)\operatorname{Log}[x1] -$

$6\ x1^3\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 18\ x1^4\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$18\ x1^5\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 12\ x1^6\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] +$

$18\ x1^3\ y1\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] - 54\ x1^4\ y1\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] +$

$54\ x1^5\ y1\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] - 36\ x1^6\ y1\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$18\ x1^3\ y1^2\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 54\ x1^4\ y1^2\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$54\ x1^5\ y1^2\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 36\ x1^6\ y1^2\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] +$

$6\ x1^3\ y1^3\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] - 18\ x1^4\ y1^3\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] +$

$18\ x1^5\ y1^3\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] - 12\ x1^6\ y1^3\operatorname{Log}\Big[-\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$6\ x1^6\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 18\ x1^6\ y1\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$18\ x1^6\ y1^2\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 6\ x1^3\ y1^3\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big] -$

$18\ x1^4\ y1^3\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big] + 18\ x1^5\ y1^3\operatorname{Log}\Big[\dfrac{(-1+x1)\ y1}{-1+y1}\Big]\Big)\Big)$

After the fifth integration with respect to y1, we get a special function, the dilogarithm $\operatorname{Li}_2(x)$.

In[22]:= **fastIntegrate[%, regions[[1, -5]]] // Simplify**

Out[22]= $\dfrac{1}{864\ x1^3}\ (120\ i\ \pi - 120\ x1 - 180\ i\ \pi\ x1 + 120\ x1^2 + 180\ i\ \pi\ x1^2 - 130\ x1^3 - 660\ i\ \pi\ x1^3 +$

$180\ \pi^2\ x1^3 + 960\ x1^4 + 900\ i\ \pi\ x1^4 - 540\ \pi^2\ x1^4 - 1419\ x1^5 - 360\ i\ \pi\ x1^5 +$

$540\ \pi^2\ x1^5 + 591\ x1^6 - 180\ \pi^2\ x1^6 + 180\ (-1 + x1)^3\ x1^3\operatorname{Log}[-1+x1]^2 -$

$360\ i\ \pi\ x1^3\operatorname{Log}[x1] + 1080\ i\ \pi\ x1^4\operatorname{Log}[x1] - 1080\ i\ \pi\ x1^5\operatorname{Log}[x1] +$

$360\ i\ \pi\ x1^6\operatorname{Log}[x1] - 360\ x1^3\operatorname{Log}[1-x1]\operatorname{Log}[x1] + 1080\ x1^4\operatorname{Log}[1-x1]\operatorname{Log}[x1] -$

$1080\ x1^5\operatorname{Log}[1-x1]\operatorname{Log}[x1] + 360\ x1^6\operatorname{Log}[1-x1]\operatorname{Log}[x1] +$

$60\ (-1+x1)^3\operatorname{Log}[-1+x1]\ (2 + 3\ x1 + 6\ x1^2 - 6\ i\ \pi\ x1^3 - 6\ x1^3\operatorname{Log}[x1]) +$

$360\ (-1+x1)^3\ x1^3\operatorname{PolyLog}[2, x1])$

The sixth integration with respect to x1 results in the final result for the first region, a numeric quantity containing π, $\ln(2)$, and $\ln^2(2)$.

In[23]:= **fastIntegrate[%, regions[[1, -6]]]**

Out[23]= $\dfrac{-1771 + 170\ \pi^2 + 800\operatorname{Log}[2] - 960\operatorname{Log}[2]^2}{18432}$

All of the last six integrations get carried out recursively using our above-defined function `multiDimensionalInte`, grate.

In[24]:= **multiDimensionalIntegrate[area, Sequence @@ regions[[1]]]**

Out[24]= $\dfrac{-1771 + 170\,\pi^2 + 800\,\text{Log}[2] - 960\,\text{Log}[2]^2}{18432}$

Now, we carry out all of the 216 multidimensional integrations. We take the real part because we expanded logarithm in intermediate steps. This could potentially introduce imaginary parts in the form $i\,k\,\pi$. This calculation will take a few minutes.

In[25]:= **Simplify[#, TransformationFunctions -> {Automatic,**
**   (# /. Log[x_] :> Log[2, x]/Log[2])&}]& @**
**   (Re[Together[Plus @@**
**   Apply[multiDimensionalIntegrate[area, ##]&, regions, {1}]]] // Timing)**

Out[25]= $\left\{1017.46 \text{ Second}, \dfrac{11}{288}\right\}$

All π and log(2) terms cancelled, and we got (taking into account the triangles with negative orientation) for the expectation value, the simple result $A = 11/144$.

The degree of difficulty to do multidimensional integrals is often depending sensitively from the order of the integration. As a check of the last result and for comparison, we now first evaluate the three integrations over the y_i and then the three integration over the x_i. For this situation, we have only 62 six-dimensional regions.

In[26]:= **cad2 = GenericCylindricalAlgebraicDecomposition[**
**   signedTriangleArea && unitCube6D,**
**   {x1, x2, x3, y1, y2, y3}];**

regions2 = Apply[List, Apply[{#3, #1, #5} &,
**   cad2[[1]] //. a_ && (b_ || c_) :> a && b || a && c, {2}], {0,**

Length[regions2]

Out[30]= 62

And doing the integrations and simplifying the result takes now only a few seconds. Again, we obtain the result 11/288.

In[31]:= **Simplify[Together[Re[Plus @@**
**   Apply[multiDimensionalIntegrate[area, ##]&, regions2, {1}]]],**
**   TransformationFunctions ->**
**   {(# /. Log[k_Integer] :> (Plus @@ ((#2 Log[#1])& @@@**
**   FactorInteger[k])))&}] // Timing**

Out[31]= $\left\{24.435 \text{ Second}, \dfrac{11}{288}\right\}$

Using numerical integration, we can calculate an approximative value of this integral to support the result $11/144$.

In[32]:= **(SeedRandom[111];**
**   NIntegrate[Evaluate[Abs[area]], {x1, 0, 1}, {y1, 0, 1},**
**   {x2, 0, 1}, {y2, 0, 1}, {x3, 0, 1}, {y3, 0, 1},**
**   Method -> QuasiMonteCarlo, MaxPoints -> 10^6,**
**   PrecisionGoal -> 3])**

Out[32]= 0.0763889

This result confirms the above result.

In[33]:= **N[2 %%[[2]]]**

Out[33]= 0.0763889

We could now go on and calculate the probability distribution for the areas. The six-dimensional integral to be calculated is now

$$p(A) \sim \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \delta(A - \mathcal{A}(x_1, x_2, x_3, y_1, y_2, y_3)) \, dy_3 \, dx_3 \, dy_2 \, dx_2 \, dy_1 \, dx_1 \, ,$$

$$\mathcal{A}(x_1, x_2, x_3, y_1, y_2, y_3) = |x_3 \, y_1 - x_2 \, y_1 + x_1 \, y_2 - x_3 \, y_2 + x_2 \, y_3 - x_1 \, y_3|.$$

(Here we temporarily changed $A \to 2 A$ so that all variables involved range over the interval [0, 1].

This time, before subdividing the integration variable space into subregions, we carry out the integral over y_3 to eliminate the Dirac δ function. To do this, we use the identity

$$\int_a^b \delta(y - f(x)) \, dx = \int_a^b \sum_k \frac{\delta(y - x_{0,k})}{|f'(x_{0,k})|} \, dx$$

where the $x_{0,k}$ are the zeros of $f(x)$ in $[a, b]$.

Expressing y_3 through x_1, x_2, x_3, y_1, y_2, and A yields the following expression.

In[34]:= **soly3 = Solve[A ==** (* or − *) **(-x2 y1 + x3 y1 + x1 y2 − x3 y2 −**
                         x1 y3 + x2 y3), y3][[1, 1, 2]]

Out[34]= $\dfrac{-A - x2\ y1 + x3\ y1 + x1\ y2 - x3\ y2}{x1 - x2}$

And the derivative from the denominator becomes $|x_1 - x_2|$.

In[35]:= **D[-x2 y1 + x3 y1 + x1 y2 − x3 y2 − x1 y3 + x2 y3, y3]**

Out[35]= − x1 + x2

Now is a good time to obtain a decomposition of the space into subregions. In addition to the constraints following from the geometric constraints of the integration variables being from the unit square, we add three more inequalities: 1) $0 < y_3(x_1, x_2, x_3, y_1, y_2; A) < 1$ to ensure the existence of a zero inside the Dirac δ function argument; 2) $A > 0$ for positive oriented areas; and 3) $x_1 > x_2$ to avoid the absolute value in the denominator (the case $x_1 < x_2$ follows from symmetry).

In[36]:= **cad = Experimental`GenericCylindricalAlgebraicDecomposition[**
            0 < soly3 < 1 && A > 0 && x1 > (* or < *) **x2 &&**
            0 < x1 < 1 && 0 < x2 < 1 &&
            0 < x3 < 1 && 0 < y2 < 1 && 0 < y1 < 1,
        {A, x1, x2, x3, y1, y2}];

This time, we get a total of 1282 subregions.

In[37]:= **(l1 = cad[[1]] //. a_ && (b_ || c_) :> (a && b) || (a && c)) // Length**

Out[37]= 1282

One expects the probability distribution $p(A)$ to be a piecewise smooth function of λ. Six λ-interval arise naturally from the decomposition.

In[38]:= **Union[First /@ l1]**

Out[38]= $0 < A < \frac{1}{6}$ || $\frac{1}{6} < A < \frac{1}{5}$ || $\frac{1}{5} < A < \frac{1}{4}$ || $\frac{1}{4} < A < \frac{1}{3}$ || $\frac{1}{3} < A < \frac{1}{2}$ || $\frac{1}{2} < A < 1$

In[39]:= **ASortedRegions = {#[[1, 1, 2]] < A < #[[1, 1, 3]] , Rest /@ #}& /@**
Split[Sort[(# /. Inequality[a_, Less, b_, Less, c_] :>
{b, a, c} /. And -> List)& /@ (List @@ l1)], #1[[1]] === #2[[1]]&]

Here is the number of regions for the six λ-intervals.

In[40]:= **{#1, Length[#2] "subregions"}& @@@ ASortedRegions**

Out[40]= $\left\{\left\{0 < A < \frac{1}{6}, 317 \text{ subregions}\right\}, \left\{\frac{1}{6} < A < \frac{1}{5}, 324 \text{ subregions}\right\},\right.$

$\left\{\frac{1}{5} < A < \frac{1}{4}, 310 \text{ subregions}\right\}, \left\{\frac{1}{4} < A < \frac{1}{3}, 216 \text{ subregions}\right\},$

$\left.\left\{\frac{1}{3} < A < \frac{1}{2}, 99 \text{ subregions}\right\}, \left\{\frac{1}{2} < A < 1, 16 \text{ subregions}\right\}\right\}$

The regions themselves look quite similar to the above ones.

In[41]:= **{#1, #2[[1]]}& @@@ ASortedRegions**

Out[41]= $\left\{\left\{0 < A < \frac{1}{6}, \{\{x1, 0, A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.\right.$

$\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\},$

$\left\{\frac{1}{6} < A < \frac{1}{5}, \{\{x1, 0, A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.$

$\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\},$

$\left\{\frac{1}{5} < A < \frac{1}{4}, \{\{x1, 0, A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.$

$\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\},$

$\left\{\frac{1}{4} < A < \frac{1}{3}, \{\{x1, 0, A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.$

$\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\},$

$\left\{\frac{1}{3} < A < \frac{1}{2}, \{\{x1, 0, A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.$

$\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\},$

$\left\{\frac{1}{2} < A < 1, \{\{x1, 0, 1 - A\}, \{x2, 0, x1\}, \{x3, A + x1, 1\},\right.$

$\left.\left.\left\{y1, -\frac{A}{x2 - x3}, \frac{-A - x1 + x2}{x2 - x3}\right\}, \left\{y2, 0, \frac{-A - x2\,y1 + x3\,y1}{-x1 + x3}\right\}\right\}\right\}\right\}$

To carry out the remaining five-dimensional integration, we could use a function like `x1x2x3y1y2Integration`. The optional functions $\mathcal{S}$ and $\mathcal{T}$ could carry out simplifications of the integrands to ease the work of `Integrate` and the results of `Integrate` to ease the work of `Limit`.

```
In[42]:= x1x2x3y1y2Integration[region_, startIntegrand:i0_,
                               𝒮_:Identity, 𝒯_:Identity] :=
     Module[{𝒟ℒ, y2y1x3x2x1Regions},
        (* form difference of integral at upper limit minus
           integral at lower limit *)
        𝒟ℒ = (Limit[#1, #2[[1]] -> #2[[3]], Direction -> +1] -
              Limit[#1, #2[[1]] -> #2[[2]], Direction -> -1])&;
        (* integration regions *)
        y2y1x3x2x1Regions = Reverse[region];
        (* carry out the five integrations *)
        Fold[𝒟ℒ[𝒯[Integrate[𝒮[#1]], #2[[1]]]], #2]&, i0,
           y2y1x3x2x1Regions]]
```

Here is a typical result for one of the 1282 five-dimensional integrals.

```
In[43]:= (* no special simplifiers needed for the subregion {5, 40} *)
     x1x2x3y1y2Integration[ASortedRegions[[5, 2, 40]], 1/(x1 - x2)] //
                                               Simplify[#, A > 0]&
```

$$Out[44]=\ \frac{1}{8}\ (4\,A^2\,(-3+\text{Log}[16])+(-1+A)\,(-5+9\,A+2\,(1+A)\,\text{Log}[1-A]-2\,(1+A)\,\text{Log}[A]))$$

Naturally, we will start carrying out the integrations for the area interval $1/2 < A < 1$, which has the smallest number of regions.

```
In[45]:= Module[{j = 6, sumA = 0, res},
        Do[res = x1x2x3y1y2Integration[ASortedRegions[[j, 2, k]], 1/(x1 - x2)];
           sumA = sumA + res;
           If[MemberQ[res, _Integrate, {0, Infinity}], Print[{k, res}]],
           {k, Length[ASortedRegions[[j, 2]]]}];
        sumA];
```

The result is a nice short expression in the area *A*. Because it contains polylogarithms, we simplify it using `FullSimplify` (which simplifies special functions).

```
In[46]:= pA = % // FullSimplify[#, ASortedRegions[[6, 1]]]&
```

$$Out[46]=\ \frac{1}{12}\ \Big(18-A\,(18+(12+7\,A)\,\pi^2)+15\,A^2\,(-6+\text{Log}[A])\,\text{Log}[A]+$$
$$6\,\text{Log}[1-A]\,(3\,(-1+A)\,(1+5\,A)+A^2\,\text{Log}[A])-$$
$$6\,A^2\,\text{PolyLog}\Big[2,\frac{-1+A}{A}\Big]+6\,A\,(12+7\,A)\,\text{PolyLog}[2,A]\Big)$$

Applying an identity for the dilogarithm allows obtaining the following concise result for the probability of the area (in this step, we restore the original units of *A*).

$$p(A)=$$
$$-4\,(-12\,(\ln(2\,A)-5)\,\ln(2\,A)\,A^2-24\,(A+1)\,\text{Li}_2(2\,A)\,A+4\,(A+1)\,\pi^2\,A+6\,A+(12\,(2-5\,A)\,A+3)\,\ln(1-2\,A)-3)$$

```
In[47]:= polyLogRule = PolyLog[2, (A - 1)/A] ->
                       PolyLog[2, A] + Log[1 - A] Log[A] - Log[A]^2/2 - Pi^2/6;
```

```
In[48]:= p[A_] = 8 (pA /. polyLogRule) /. A -> 2A  // FullSimplify[#, 0 < A < 1]&
```

$$Out[48]=\ -4\,(-3+6\,A+4\,A\,(1+A)\,\pi^2+(3+12\,(2-5\,A)\,A)\,\text{Log}[1-2\,A]-$$
$$12\,A^2\,(-5+\text{Log}[2\,A])\,\text{Log}[2\,A]-24\,A\,(1+A)\,\text{PolyLog}[2,2\,A])$$

Using the other five intervals for *A* would give the same result, but only after more work because of the substantially larger number of regions.

Here is a quick check of the normalization and the average area. Now, it is also straightforward to calculate the variance.

```
In[49]:= {Integrate[p[A], {A, 0, 1/2}], Integrate[A p[A], {A, 0, 1/2}],
         Integrate[(A - 11/144)^2 p[A], {A, 0, 1/2}]}
```

$$Out[49]= \left\{1, \frac{11}{144}, \frac{95}{20736}\right\}$$

Interestingly, the probability distribution is the solution of a simple linear forth-order differential equation with polynomial coefficients.

```
In[50]:= (* eliminate logarithms and polylogarithms *)
         GroebnerBasis[Numerator[Together[
                   Table[dp[k] - D[p[A], {A, k}], {k, 0, 4}]]], {},
                   Reverse @ Union[Cases[p[A], _Log | _PolyLog, Infinity]],
                   Sort -> True] // Factor
```

$$Out[51]= \{A \, (768 \, dp[1] - 384 \, dp[2] - 768 \, A \, dp[2] - 384 \, A \, dp[3] +$$
$$768 \, A^2 \, dp[3] + 4 \, A^2 \, dp[3]^2 - 8 \, A^3 \, dp[3]^2 + 4 \, A^4 \, dp[3]^2 + 4 \, A^3 \, dp[3] \, dp[4] -$$
$$12 \, A^4 \, dp[3] \, dp[4] + 8 \, A^5 \, dp[3] \, dp[4] + A^4 \, dp[4]^2 - 4 \, A^5 \, dp[4]^2 + 4 \, A^6 \, dp[4]^2),$$
$$9216 - 18432 \, A - 768 \, dp[0] + 384 \, A \, dp[2] + 384 \, A^2 \, dp[2] + 480 \, A^2 \, dp[3] -$$
$$768 \, A^3 \, dp[3] - 4 \, A^3 \, dp[3]^2 + 8 \, A^4 \, dp[3]^2 - 4 \, A^5 \, dp[3]^2 +$$
$$96 \, A^3 \, dp[4] - 192 \, A^4 \, dp[4] - 4 \, A^4 \, dp[3] \, dp[4] + 12 \, A^5 \, dp[3] \, dp[4] -$$
$$8 \, A^6 \, dp[3] \, dp[4] - A^5 \, dp[4]^2 + 4 \, A^6 \, dp[4]^2 - 4 \, A^7 \, dp[4]^2\}$$

```
In[52]:= Sort[(* find short differential equation *)
         GroebnerBasis[%, {dp[4], dp[3], dp[2], dp[1], dp[0]}],
         (Length[#1] < Length[#2])&][[1]]
```

$$Out[52]= -96 + 192 \, A + 8 \, dp[0] - 8 \, A \, dp[1] + 4 \, A^2 \, dp[2] - A^2 \, dp[3] - A^3 \, dp[4] + 2 \, A^4 \, dp[4]$$

Here is a quick check of this differential equation.

```
In[53]:= A^3 (1 - 2 A) p''''[A] + A^2 p'''[A] - 4 A^2 p''[A] + 8 A p'[A] -
                                    8 p[A] - 96 (2 A - 1) // Simplify
```

$$Out[53]= 0$$

We end by a comparing the calculated density with a modeled one. The function `modelData` generates *o* random triangles in the unit square and returns the histogram data for *bins* bins.

```
In[54]:= modelData[o_, bins_] :=
    MapIndexed[(* scale *) {#2[[1]]/bins/2, 2 bins #1/o}&, #]& @
      Compile[{}, Module[{T = Table[0, {bins}], A,
                    p1, p2, p3, d1x, d1y, d2x, d2y, cA},
                  Do[(* the random points *)
                      {p1, p2, p3} = Table[Random[], {3}, {2}];
                      {d1x, d1y} = p3 - p1; {d2x, d2y} = p2 - p1;
                      (* area of the triangle *)
                      A = Abs[d1x d2y - d1y d2x]/2;
                      (* discretized area *)
                      cA = Ceiling[bins A/(1/2)];
                      (* increase area counters *)
                      T[[cA]] = T[[cA]] + 1, {i, o}];
                      (* return counts *) T]][]
```

Using 10 million random triangles and 100 bins, we obtain an excellent agreement between the modeled data and the theoretical probability density. The right graphic shows the weighted density *A p(A)*.

In[55]:= **Show[GraphicsArray[**
    Block[{$DisplayFunction = Identity},
     {(* probability density *)
      Plot[𝑝[A], {A, 0, 1/2}, PlotRange -> All, PlotStyle -> {Hue[0]},
         (* modeled probabilities *)
         Prolog -> {PointSize[0.01], GrayLevel[0],
                Point /@ modelData[10^6, 100]}, PlotRange -> All],
     (* weighted probability density *)
     Plot[A 𝑝[A], {A, 0, 1/2}, PlotRange -> All]}]]]
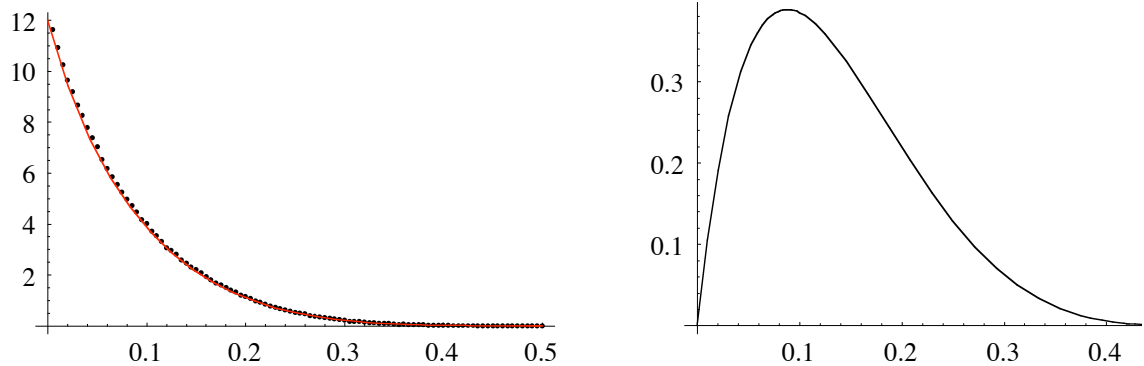
Out[55]= **- GraphicsArray -**

Having calculated and modeled the probability distribution of the area, a natural problem arising is the form of the probability density $p(x, y)$ of a point $\{x, y\}$ being inside a randomly chosen triangle. Geometrically "inside" means that the point $\{x, y\}$ lies in the three left (or right) half planes generated through the three triangle edges (assuming the three triangle edges are uniformly oriented). This means the integral to be evaluated for the probability density is

$$p(x, y) = 2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta_{2,1}(x, y)\,\theta_{3,3}(x, y)\,\theta_{1,3}(x, y)\,dy_3\ dx_3\ dy_2\ dx_2\ dy_1\ dx_1\ ,$$
$$\theta_{i,j}(x, y) = \theta((p_i - p_j).(p - p_j))$$

where $p_i = \{x_i, y_i\}$ are the triangle vertices and $p = \{x, y\}$. (The factor 2 accounts for the orientation of the triangles).

In[56]:= *θθθ* **= Module[{p1 = {x1, y1}, p2 = {x2, y2}, p3 = {x3, y3},**
               p = {x, y}, normal = (Reverse[#]{1, -1})&},
               (* point p is to the left of each edge *)
               normal[p2 - p1].(p - p1) > 0 &&
               normal[p3 - p2].(p - p2) > 0 &&
               normal[p1 - p3].(p - p3) > 0]

Out[56]= $(x1 - x2)\ (y - y1) + (x - x1)\ (-y1 + y2) > 0$ &&
    $(x2 - x3)\ (y - y2) + (x - x2)\ (-y2 + y3) > 0$ &&
    $(-x1 + x3)\ (y - y3) + (x - x3)\ (y1 - y3) > 0$

To calculate the sixfold integral, we will follow the already twice successfully-used strategy to first calculate a decomposition of the integration domain. Because of the obvious fourfold rotational symmetry of $p(x, y)$ around the square center $\{1/2, 1/2\}$, we use the restrictions $1/2 < x < 1$, $1/2 < y < x$ to reduce the number of cells returned. We obtain 327 cells (without using the symmetry, we would obtain 3738 cells).

```
In[57]:= cadxy = Experimental`GenericCylindricalAlgebraicDecomposition[
              000 && 0 < x1 < 1 && 0 < x2 < 1 && 0 < x3 < 1 &&
                  0 < y1 < 1 && 0 < y2 < 1 && 0 < y3 < 1 &&
                  1/2 < x < 1 && 1/2 < y < x,
             {x, y, x1, x2, x3, y1, y2, y3}];
```

```
In[58]:= l1 = cadxy[[1]] //. a_ && (b_ || c_) :> a && b || a && c;
```

```
In[59]:= Length[l1]
```

Out[59]= 327

All cells span the specified *x,y*-domain. This means, the density $p(x, y)$ is continuous within this domain.

```
In[60]:= Union[Take[#, 2]& /@ l1]
```

Out[60]= $\frac{1}{2} < x < 1$ && $\frac{1}{2} < y < x$

The cells of the 6D integration domain have similar-looking boundaries as the cells from the above calculations.

```
In[61]:= xyRegions = (# /. Inequality[a_, Less, b_, Less, c_] :>
                          {b, a, c} /. And -> List)& /@
                ((* remove x and y parts *) List @@ Drop[#, 2]& /@ l1);
```

```
In[62]:= xyRegions[[1]]
```

Out[62]= $\left\{\left\{x1, 0, \frac{-x + y}{-1 + y}\right\}, \{x2, 0, x1\}, \{x3, x, 1\}, \left\{y1, 0, \frac{-x1\,y + x2\,y}{-x + x2}\right\},\right.$
$\left.\{y2, 0, 1\}, \left\{y3, \frac{x2\,y - x3\,y - x\,y2 + x3\,y2}{-x + x2}, \frac{x1\,y - x3\,y - x\,y1 + x3\,y1}{-x + x1}\right\}\right\}$

The values of the integrals are slightly simpler looking than in the last calculation—no polylogarithms occur in the result.

```
In[63]:= x1x2x3y1y2Integration[xyRegions[[-1]], 1]
```

Out[63]= $-\frac{(-1 + x)^3\,(-1 + y)^4\,\text{Log}[1 - x]}{12\,y^2} + \frac{1}{72\,y^8}\left((-1 + x)^3\,(-1 + y)^3\right.$
$\left.\left(-1 + 4\,y - 3\,y^2 - y^3 - 2\,y^4 - 6\,y^5 + 12\,y^6 + 6\,(-1 + y)\,y^6\,\text{Log}\left[-\frac{(-1 + x)\,(-1 + y)^2}{y^2}\right]\right)\right)$

We now carry out the 327 6D integrals. This takes a few minutes.

```
In[64]:= (integrals =
          Table[res = x1x2x3y1y2Integration[xyRegions[[k]], 2];
                If[MemberQ[res, _Integrate, {0, Infinity}], Print[{k, res}]];
                res, {k, Length[xyRegions]}]); // Timing
```

Out[64]= {980.54 Second, Null}

The unsimplified form of the result is a large expression.

```
In[65]:= sum1 = Expand[Plus @@ integrals];
         Length[sum1]
```

Out[66]= 7561

Here are the first and the last terms of sum1.

In[67]:= `{Take[sum1, 6], Take[sum1, -6]}`

Out[67]= $\left\{ \dfrac{5213}{72} + \dfrac{163}{12\,(1-x)} + \dfrac{215}{8\,(-1+x)} + \dfrac{17}{2\,x} + \dfrac{26}{3\,(-1+x)\,x} - \dfrac{7097\,x}{24} , \right.$

$\dfrac{x\,y^4\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{2\,(-1+y)} - \dfrac{x^2\,y^4\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{-1+y} +$

$\dfrac{x^3\,y^4\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{2\,(-1+y)} - \dfrac{3\,x\,y^5\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{2\,(-1+y)^2} +$

$\left. \dfrac{3\,x^2\,y^5\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{(-1+y)^2} - \dfrac{3\,x^3\,y^5\,\text{Log}\left[\frac{y\,(-1+x+2\,y-2\,x\,y-y^2+x\,y^2)}{(-1+y)^3}\right]}{2\,(-1+y)^2} \right\}$

We see various logarithms with relatively complicated arguments. In `sum1` more than 60 different logarithms occur.

In[68]:= `Cases[sum1, _Log, Infinity] // Union // Length`

Out[68]= 63

We simplify the logarithms by power expanding them. This will result in some spurious imaginary parts which we remove by taking the real part.

In[69]:= `sum2 = sum1 //. Log[ξ_] :> PowerExpand[Log[Factor[ξ]]];`

Now, we have only a handful of different logarithms left.

In[70]:= `Cases[sum2, _Log, Infinity] // Union`

Out[70]= `{Log[1 - x], Log[-1 + x], Log[x], Log[-1 + y], Log[y]}`

In[71]:= `sum3 = ComplexExpand[Re[sum2]];`

Simplifying `sum3` gives the following final result for the probability density of the point $\{x, y\}$ being inside a randomly chose triangle.

$$p(x, y) = \frac{1}{2}\,\frac{(x-1)^2}{x\,y\,(y-1)}$$
$$(12\,x^2\,y^4 - 30\,x\,y^4 + 3\,y^4 - 24\,x^2\,y^3 + 60\,x\,y^3 - 6\,y^3 + 25\,x^2\,y^2 - 43\,x\,y^2 + 4\,y^2 - 13\,x^2\,y + 13\,x\,y - y + x^2 - x) -$$
$$\frac{3}{2}\,(x-1)^2\,(6\,x\,y^2 - 4\,y^2 - 6\,x\,y + 4\,y + 2\,x - 1)\,\log\!\left(\frac{1}{x} - 1\right) +$$
$$\frac{3}{2}\,(x-1)^2\,(2\,y-1)\,(2\,x\,y^2 - 2\,y^2 - 2\,x\,y + 2\,y + 2\,x - 1)\,\log\!\left(\frac{1}{y} - 1\right)$$

In[72]:= `p[x_, y_] = FullSimplify[sum3, 1/2 < x < 1 && 1/2 < y < x] //`
                                          `Collect[#, _Log, Factor]&`

Out[72]= $\dfrac{1}{2\,x\,(-1+y)\,y}$

$((-1+x)^2\,(-x+x^2-y+13\,x\,y-13\,x^2\,y+4\,y^2-43\,x\,y^2+25\,x^2\,y^2-6\,y^3+60\,x\,y^3 -$
$24\,x^2\,y^3 + 3\,y^4 - 30\,x\,y^4 + 12\,x^2\,y^4)) -$

$\dfrac{3}{2}\,(-1+x)^2\,(-1+2\,x+4\,y-6\,x\,y-4\,y^2+6\,x\,y^2)\,\text{Log}\!\left[-1+\dfrac{1}{x}\right] +$

$\dfrac{3}{2}\,(-1+x)^2\,(-1+2\,y)\,(-1+2\,x+2\,y-2\,x\,y-2\,y^2+2\,x\,y^2)\,\text{Log}\!\left[-1+\dfrac{1}{y}\right]$

Here is a quick check that the power expanding did indeed not change the numerical value of the density.

In[73]:= `{sum1, p[x, y]} /. x -> 2/E /. y -> 2/Pi // N[#, 10]&`

Out[73]= $\{0.13621111723 + 0. \times 10^{-12}\ i, 0.1362111172\}$

The probability that the center of the square is inside a randomly chosen triangle is exactly 25%.

In[74]:= `p[1/2, 1/2]`

Out[74]= $\dfrac{1}{4}$

The next graphic shows the calculated probability density on the left and a modeled one on the right. The two densities look identical.

In[75]:=
```
Show[GraphicsArray[
   Block[{$DisplayFunction = Identity},
   {(* exact probability *)
    Module[{ε = 10^-12, pp = 20, polys},
    (* plot in the domain 1/2 < x < 1 ∧ 1/2 < y < x *)
    polys = Map[(# - {1/2, 1/2, 0})&, Cases[
       ParametricPlot3D[Evaluate @ {x, 1/2 + η (x - 1/2),
                                   p[x, 1/2 + η (x - 1/2)]},
                    {x, 1/2, 1 - ε}, {η, 0, 1},
                         PlotPoints -> pp], _Polygon, Infinity], {-2}];
    Graphics3D[{EdgeForm[], (* generate seven other parts *)
       Map[(# + {1/2, 1/2, 0})&,
         {Apply[{#2, #1, #3}&, #, {-2}], #}&[
            {Map[(# {-1, 1, 1})&, #, {-2}], #}&[
              {Map[(# {1, -1, 1})&, #, {-2}], #}&[polys]]], {-2}]},
          BoxRatios -> {1, 1, 1/2}, Axes -> True]],
    (* modeled probability *)
    Module[{d = 60, o = 10^4, data, if},
    data = Compile[{},
    Module[{T = Table[0, {d}, {d}], p1, p2, p3, xc, yc, mp, σ},
     Do[{p1, p2, p3} = Table[Random[], {3}, {2}]; mp = (p1 + p2 + p3)/3;
         (* orientation of the normals *)
         σ = Sign[(Reverse[p2 - p1]{1, -1}).(mp - p1)];
         (* are discretized square points inside triangle? *)
         Do[If[σ (Reverse[p2 - p1]{1, -1}).({x, y} - p1) > 0 &&
             σ (Reverse[p3 - p2]{1, -1}).({x, y} - p2) > 0 &&
             σ (Reverse[p1 - p3]{1, -1}).({x, y} - p3) > 0,
             (* increase counters *)
             {xc, yc} = Round[{x, y} (d - 1)] + 1;
             T[[xc, yc]] =  T[[xc, yc]] + 1,
          {x, 0, 1, 1/(d - 1)}, {y, 0, 1, 1/(d - 1)}], {o}]; T]][];
    (* interpolated scaled counts *)
    if = Interpolation[Flatten[MapIndexed[Flatten[
            {(#2 - {1, 1})/(d - 1), #1}]&, data/o, {2}], 1]];
    (* interpolated observed frequencies *)
    Plot3D[if[x, y], {x, 0, 1}, {y, 0, 1}, Mesh -> False]}]]]]
```
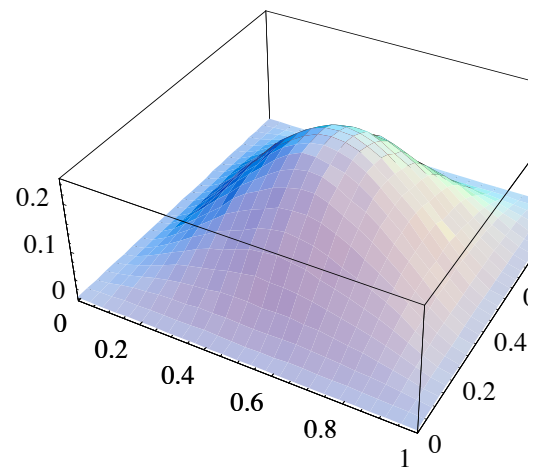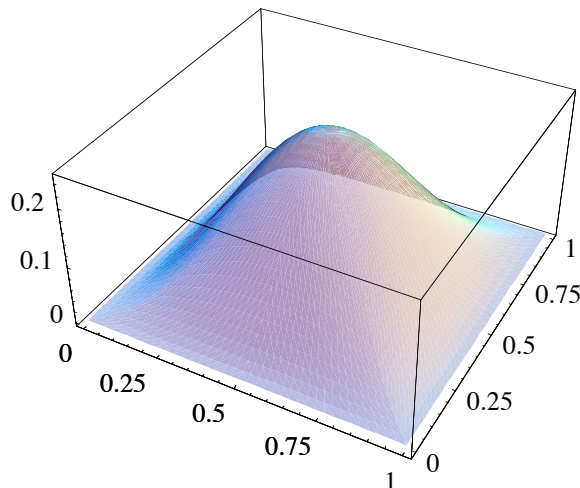

Out[75]= - GraphicsArray -

We end by integrating the calculated probability density $p(x, y)$ over the unit square. $p(x, y)$ is the probability that the point $\{x, y\}$ is inside a randomly chosen triangle. This means the average of $p(x, y)$ is again the area of a randomly chosen triangle, namely $11/144$.

In[76]:= **8 Integrate[p[x, y], {x, 1/2, 1}, {y, 1/2, x}]**

Out[76]= $\dfrac{11}{144}$

For a similar probabilistic problem, the Heilbronn triangle problem, see [945★].

In[77]:= **Σ (* session summary *) TMGBs`PrintSessionSummary[]**

◦ **Session Summary:** (Evaluated with *Mathematica* 5.1)

Inputs evaluated:	76
CPU time used:	2550 s
Wall clock time elapsed:	2577 s
Max. memory used:	508 MB
Variables in use:	143

■ 1.10.2 $\cos\left(\frac{2\pi}{257}\right)$ à la Gauss

In the early morning of March 29 in 1796, Carl Friedrich Gauss (while still in bed) recognized how it is possible to construct a regular 17-gon by ruler and compass; or more arithmetically and less geometrically speaking, he expressed $\cos\left(\frac{2\pi}{17}\right)$ in terms of square roots and the four basic arithmetic operations of addition, subtraction, multiplication, and division only. (This discovery was the reason why he decided to become a mathematician [1483★], [709★], [1808★].) His method works immediately for all primes of the form $2^{2^j} + 1$, so-called Fermat numbers F_j [1091★]. For $j = 0$ to 4, we get the numbers 3, 5, 17, 257, and 65537. ($j = 5, \ldots, 14$ do not give primes; we return to this at the end of this section.) The problem to be solved is to express the roots of $z^p = 1$, where p is a Fermat prime in square roots. One obvious solution of this equation is $z = 1$. After dividing $z^p = 1$ by this solution, we get as the new equation to be solved:

$$z^{p-1} + z^{p-2} + \cdots + z + 1 = 0.$$

It can be shown that there are no further rational zeros; so this equation cannot be simplified further in an easy way. Let us denote (by following Gauss's notation here and in the following) the solution $\exp\left(\frac{\lambda 2\pi i}{p}\right)$, λ integer, $1 \le \lambda \le p - 1$ of this equation by $\overline{\lambda}$ (which is, of course, a solution, but which contains a pth root). Gauss's idea, which solves the above equation exclusively in square roots, is to group the roots of the above equation in a recursive way such that the explicit values of the sums of these roots can be expressed in numbers and square roots. Each step then rearranges these roots until finally only groups of length two remain. These last groups are then just of the form $\cos\left(\frac{j 2\pi}{p}\right)$.

Let us describe this idea in more detail. First, we need the number-theoretic notion of a primitive root: the number g is called a primitive root of p if the set of numbers $\{g^i \bmod p\}_{i=0,\ldots,p-2}$ are (up to their order) identical with the numbers 1, ..., $p - 1$. The following input shows that the number 3 is a primitive root of 5, 17, 257, and 65357. (To avoid the calculation of huge numbers caused by exponentiation, we use `PowerMod[g, i, n]`, which is mathematically equivalent to `Mod[g^i, n]`, but much faster.)

In[1]:= **Function[p, Sort[Array[PowerMod[3, #, p]&, p - 1, 0]] ==**
Range[p - 1]] /@ {5, 17, 257, 65537}

Out[1]= {True, True, True, True}

(Only integers *n* of the form 2, 4, p^j, $2 p^j$, where *p* is an odd prime and $j > 0$ have primitive roots; they have $\phi(\phi(n))$ different ones; $\phi(n)$ is Euler's totient function.) The order of the integers in `Array[PowerMod[`*base*`, #, `*prime*`]&, p - 1, 0]` exhibits some interesting symmetry, as can be visualized for the *prime* = 257 case with the following input.

```
In[2]:= primitiveRootsGraphics[b_] := Graphics[
         {Thickness[0.002], Line[Append[#, First[#]]&[
        (* connect numbers in their permuted order *)
        {Cos[#], Sin[#]}& /@ N[2Pi Array[
            PowerMod[b, #, 257]&, 256, 0]/257]]]},
          PlotRange -> All, AspectRatio -> Automatic]
```

```
In[3]:= (* reduced residue system exists for the following 128 numbers *)
        rssNumbers = Flatten[Position[Table[Sort[Array[
                      PowerMod[i, #, 257]&, 256, 0]] ==
                       Range[256], {i, 256}], True]];
```

```
In[5]:= (* visualizations of the powermod sequences *)
        Function[bs, Show[GraphicsArray[Function[b,
          primitiveRootsGraphics[b]] /@ bs]]] /@ (* display nine examples *)
               Partition[rssNumbers[[{1, 2, 3, 33, 42, 43, 66, 106, 114}]], 3]
```
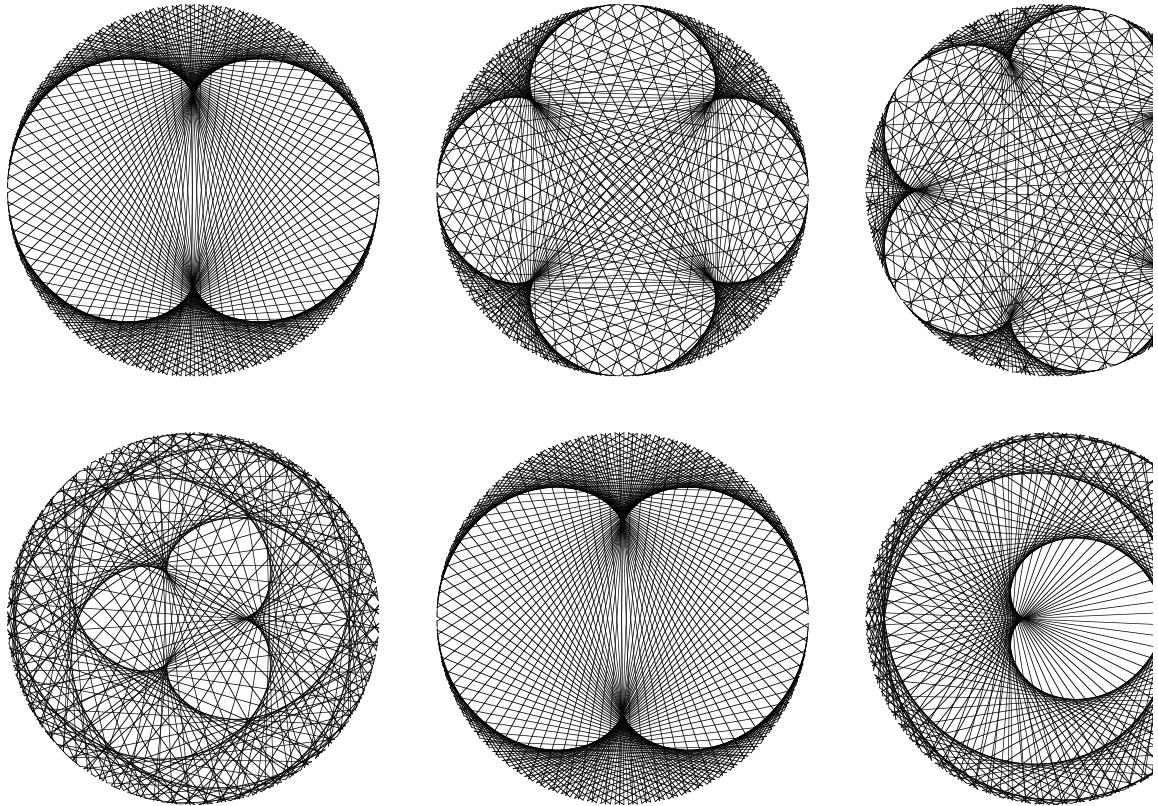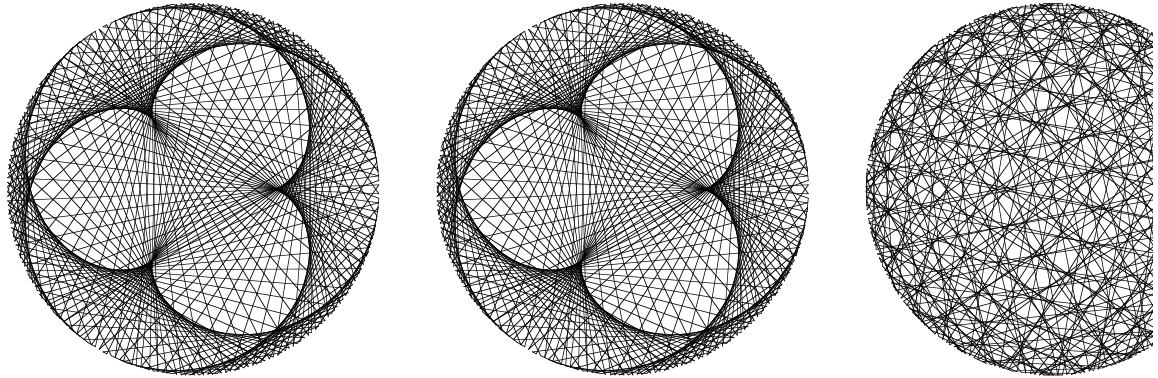
Out[6]= {- GraphicsArray -, - GraphicsArray -, - GraphicsArray -}

```
Show[primitiveRootsGraphics[#]]& /@ rssNumbers
```

(For some interesting discussions about the number of crossings and the number of regions in such pictures, see [1439✶].)

The next concept we need is that of the so-called periods. A period $\overset{f}{\widehat{\lambda}}$ to the primitive root g, containing the root $\overline{\lambda}$ and having length f, is defined by the expression below. (The dependence on the fixed quantities p and g is suppressed.)

$$\overset{f}{\widehat{\lambda}} = \sum_{j=0}^{f-1} \overline{\lambda \, g^{\frac{j(p-1)}{f}}}$$

Because the root $\overline{\lambda + p}$ is equivalent to the root $\overline{\lambda}$, we implement the construction of the periods in the following way. (We again use `PowerMod` because of speed and denote $\overline{\lambda}$ by `R[λ]`.)

```
In[7]:= period[λ_, f_, p_, g_] := Plus @@ (R /@ Mod[Mod[λ, p] Array[
                          PowerMod[g^((p - 1)/f), #, p]&, f, 0], p])
```

Let us look at two examples for the prime 17 and the primitive root 3.

```
In[8]:= period[1, 8, 17, 3]
```

Out[8]= R[1] + R[2] + R[4] + R[8] + R[9] + R[13] + R[15] + R[16]

```
In[9]:= period[3, 8, 17, 3]
```

Out[9]= R[3] + R[5] + R[6] + R[7] + R[10] + R[11] + R[12] + R[14]

We see that their sum just gives the sum of all roots. This is always the case if p is a Fermat prime; here, the case $p = 257$ is checked.

```
In[10]:= period[1, 128, 257, 3] + period[3, 128, 257, 3] == Plus @@ Array[R, 256]
```

Out[10]= True

```
In[11]:= period[5, 128, 257, 3] + period[9, 128, 257, 3] == Plus @@ Array[R, 256]
```

Out[11]= True

Dividing the last period again into subperiods by using the above definition for the periods, we find that the period `period[3, 8, 17, 3]` can be expressed as the sum of the following periods.

In[12]:= **period[3, 4, 17, 3]**

Out[12]= R[3] + R[5] + R[12] + R[14]

In[13]:= **period[11, 4, 17, 3]**

Out[13]= R[6] + R[7] + R[10] + R[11]

It can be shown that one can always represent a period in this way: One root of one of the new periods is identical to the old one ($\overline{\lambda}$), whereas one root of the other period is generated by the root $\overline{\left(\lambda \, g^{\frac{p-1}{f}}\right) \bmod(p-1)}$. The other roots of the two periods under consideration follow immediately from the above definition of the periods. In our example, we have this for one root of the second period: $\left(33^{\frac{17-1}{8}}\right) \bmod 16 = 11$. In doing this division process for the periods repeatedly, we end up in periods of length two. These periods are of the form $\overline{1 + p - 1}, \overline{2 + p - 2}, \ldots$, which give immediately $2\cos\left(\frac{2\pi}{p}\right), 2\cos\left(2 \times 2\,\frac{\pi}{p}\right), \ldots$. To explicitly calculate the values of the periods in square roots, we need the following theorem: The (numerical) values Λ_1, Λ_2 of two periods λ_1, λ_2 (which contain no higher roots than square roots and are to be discriminated from the periods λ_1, λ_2 themselves) obtained by splitting one period are the solutions of a quadratic equation. If Λ_1 and Λ_2 are the solutions of $\Lambda^2 + a_1 \Lambda + a_2 = 0$; by Vieta's theorem, we have $\Lambda_1 \Lambda_2 = -a_1$ and $\Lambda_1 + \Lambda_2 = a_2$. The sum of the two periods is just the period before splitting, and the (numerical) value of the starting period is -1. It is important to observe that the product of two periods of length f, obtained by splitting a period $\overset{2f}{\overline{\lambda}}$, can always be expressed as a linear combination of periods of length $2f$. The explicit formula for carrying out this multiplication of two periods is given by

$$\overset{f}{\overline{\lambda\mu}} = \overset{f}{\overline{\lambda_1\,\mu_1}} + \overset{f}{\overline{\lambda_2\,\mu_1}} + \cdots + \overset{f}{\overline{\lambda_f\,\mu_1}}$$

where

$$\overset{f}{\overline{\lambda}} = \overline{\lambda_1} + \overline{\lambda_2} + \cdots + \overline{\lambda_f} \quad \text{and} \quad \overset{f}{\overline{\mu}} = \overline{\mu_1} + \overline{\mu_2} + \cdots + \overline{\mu_f}.$$

After this multiplication, the periods on the right-hand side can then be expressed as periods of length $2f$ or as pure numbers. (For $f = \frac{p-1}{2}$, they can always be expressed as pure numbers, which ensures that we have appropriate starting values for the recursive calculation.) Here, the above two periods of length 8 of $p = 17$ (period[1, 8, 17, 3] and period[3, 8, 17, 3]; $\mu_1 = 1, \lambda_1 = 3, \lambda_2 = 5, \lambda_3 = 6, \lambda_4 = 7, \lambda_5 = 10, \lambda_6 = 11, \lambda_7 = 12, \lambda_8 = 14$) are multiplied in this manner.

In[14]:= **period[3 + 1, 8, 17, 3] + period[5 + 1, 8, 17, 3] +**
       period[6 + 1, 8, 17, 3] + period[7 + 1, 8, 17, 3] +
       period[10 + 1, 8, 17, 3] + period[11 + 1, 8, 17, 3] +
       period[12 + 1, 8, 17, 3] + period[14 + 1, 8, 17, 3] // Factor

Out[14]= 4 (R[1] + R[2] + R[3] + R[4] + R[5] + R[6] + R[7] + R[8] +
          R[9] + R[10] + R[11] + R[12] + R[13] + R[14] + R[15] + R[16])

By taking into account the original equation this obviously simplifies to -4. (The value of the period of length 16 was -1.)

The two values for the periods of length $\frac{p-1}{2}$ can be given in closed form.

In[15]:= **{1/2 (-1 + I^(((p - 1)/2)^2) Sqrt[p]),**
        1/2 (-1 - I^(((p - 1)/2)^2) Sqrt[p])};

This agrees with the direct numerical calculation, as shown here for $p = 17$.

In[16]:= **% /. p -> 17 // N**

Out[16]= {1.56155, -2.56155}

In[17]:= **{period[1, 8, 17, 3] /. (R -> (Exp[2Pi I #/17.]&)),**
period[3, 8, 17, 3] /. (R -> (Exp[2Pi I #/17.]&))} // N // Chop

Out[17]= $\{1.56155, -2.56155\}$

In the general case, the values of the periods with $f < \frac{p-1}{2}$ and their products have to be determined by the above-described recursive procedures.

By calculating the values of the periods Λ_1 and Λ_2 in this manner (up to the correspondence between periods and their numerical values), we can recursively continue splitting periods until we end at the periods of length 2.

Now, we implement the calculation of $\cos(\frac{2\pi}{257})$ via the routine GaussSolve[p, Λ]. Its arguments are the prime p and the name of the periods Λ. The eight functions introduced inside GaussSolve, λ, newλs, Timesλ, allλs, rules1, rules, Simplifyλ, and solStep, have the following meanings. (In every case, we use the primitive root $g = 3$, which is possible as we have proved above by explicit calculation for the primes p of interest here.)

λ[t, f] gives the numbers of the roots of the period of length f containing the root t (without using the head R as above).

newλs[t, f] returns a list with two roots that generate the two periods of length $f/2$ by splitting the period $\overset{f}{\hat{t}}$ of length f, which contains the root $\bar{t}$.

Timesλ[t, u, f] carries out the above-described multiplication of the two periods $\overset{f}{\hat{t}}$ and $\overset{f}{\hat{u}}$.

allλs[f] gives a list containing one root from every period of length f with $2 \le f \le \frac{p-1}{2}$.

rules1[f] gives a list of rules that replace all of the $\overset{f}{\widehat{\lambda_2}}, \overset{f}{\widehat{\lambda_3}}, \ldots, \overset{f}{\widehat{\lambda_f}}$ by $\overline{\lambda_1}$ if the $\overset{f}{\widehat{\lambda_i}}$ arise from $\overset{f}{\hat{\lambda}} = \overline{\lambda_1} + \overline{\lambda_2} + \cdots + \overline{\lambda_f}$. (This is possible because obviously the relation $\overset{f}{\widehat{\lambda_1}} = \overset{f}{\widehat{\lambda_2}} = \cdots = \overset{f}{\widehat{\lambda_f}}$ holds as a result of $\overset{f}{\hat{\lambda}} = \overline{\lambda_1} + \overline{\lambda_2} + \cdots + \overline{\lambda_f}$ (a period can be generated by any of its roots) and ensures that we have only the value of one root from every period to calculate.)

rules2[f] replaces the period containing the root $\overline{(\lambda g^{\frac{p-1}{f}}) \bmod (p-1)}$ by $\overset{2f}{\hat{\lambda}} - \overset{f}{\hat{\lambda}}$ to have only one new period introduced explicitly in solList at each subdivision.

Simplifyλ[t, u, f] simplifies the product of two periods calculated with Times1[t, u, f] using the rules rules1 and rules2.

solStep[t, f] carries out one subdivision of the period $\overset{f}{\hat{t}}$, and calculates the values of the periods $\overset{f/2}{\hat{u}}$ and $\overset{f/2}{\hat{v}}$ after splitting. For the determination of which root belongs to which period, we compare the numerical values of the periods. (This comparison could be done with much more effort symbolically, but because this has no influence on the result, we use this quick way.) Carrying out solStep[t, f] has no result, but only a side effect: The symbolic values for the new periods obtained by splitting $\overset{f}{\hat{t}}$ are added to the list solList.

The algorithm carried out by GaussSolve is quite simple: After the definition of the just-described functions and the calculation of all numerical values of the periods (which are stored in solNList), we collect in solArgs the ordered list of all periods $\overset{f}{\hat{\lambda}}$ that have to be subdivided in the form {λ, f}. Then, solStep is applied to all of the elements of solArgs. The result of GaussSolve[p, Λ] gives a list of replacement rules for the symbolic values of the periods Λ[λ, f]. We collect not only the result, but also all values for the periods, using this list, we can later also easily get $\cos(j2\pi/p)$ (j other than 1). Because for $p > 17$ the various lists of rules that are in use inside GaussSolve are quite big, we use Dispatch to accelerate their application (with the exception of the list solList, which is not used actively internally, but only serves as a container for the results).

```
In[18]:= GaussSolve[p:(3 | 5 | 17 | 257 | 65537), Λ_Symbol] :=
    Module[{g = 3, λ, newλs, Timesλ, allλs, rules1, rules2, Simplifyλ,
            solStep, solArgs, solNList, solList = {Λ[1, p - 1] -> - 1}},
    (* the λ's *)
    λ[t_, f_] := λ[t, f] = Function[γ, Mod[Mod[t, p] Array[
                    PowerMod[γ, #, p]&, f, 0], p]][g^((p - 1)/f)];
    (* newλs function definition with remembering *)
    newλs[t_, f_] := newλs[t, f] =
        {t, Mod[Mod[t, p] PowerMod[g, (p - 1)/f, p], p]};
    (* Timesλ function for λ multiplication *)
    Timesλ[t_, u_, f_] := Plus @@ (Λ[#, f]& /@ Mod[λ[u, f] + t, p]);
    (* allλs lists *)
    allλs[p - 1] = {1};
    allλs[f_] := allλs[f] = Flatten[Map[newλs[#, 2f]&, allλs[2f], {-1}]];
    (* rules1 for λ canonicalization *)
    rules1[f_] := rules1[f] =
    Dispatch[Map[Λ[#, f]&, Flatten[Function[a,
     Apply[Rule, Transpose[{Rest[a], Table[#,
       {Length[Rest[a]]}]&[First[a]]}], {1}]] /@
                (λ[#, f]& /@ allλs[f])], {-1}]];
    (* rules2 for λ eliminating one λ *)
    rules2[(p - 1)/2] = Λ[g, (p - 1)/2] -> - 1 - Λ[1, (p - 1)/2];
    rules2[f_] := rules2[f] = Dispatch[
      Λ[#[[2, 2]], f] -> Λ[#[[1]], 2f] - Λ[#[[2, 1]], f]& /@
            Map[{#, newλs[#, 2f]}&, allλs[2f], {-1}]];
    (* Simplifyλ for simplifying products of λs *)
    Simplifyλ[t_, u_, f_] := Fold[Expand[#1 //. #2]&,
        Expand[Timesλ[t, u, f] //. rules1[f]],
          rules2 /@ (f 2^Range[0, Log[2, (p - 1)/f] - 1])];
    (* solStep for period subdivision *)
    solStep[t_, f_] :=
    Module[{u, v, x1Px2, x1Tx2, sol1, sol2, sol1N, sol2N, numSol1},
    {u, v} = newλs[t, f];
     x1Px2 = Λ[t, f]; x1Tx2 = Simplifyλ[u, v, f/2];
     {sol1, sol2} = # + Sqrt[#^2 - x1Tx2]{1, -1}&[x1Px2/2];
     numSol1 = Λ[u, f/2] //. solNList;
     {sol1N, sol2N} = N[{sol1, sol2} //. solNList];
     solList = Flatten[{solList,
     If[Abs[sol1N - numSol1] < Abs[sol2N - numSol1],
        {Λ[u, f/2] -> sol1, Λ[v, f/2] -> sol2},
        {Λ[u, f/2] -> sol2, Λ[v, f/2] -> sol1}]}]; ];
    (* solNList for numerical values of the periods *)
    solNList = Dispatch[Apply[(Λ @ ##) ->
        (Plus @@ Exp[N[2Pi I λ[##]/p]])&,
          Flatten[Function[i, {#, i}& /@ allλs[i]] /@
                (2^Range[Log[2, p - 1], 1, -1]), 1], {1}]];
    (* stepArgs for period arguments *)
    stepArgs = Flatten[Function[i, {#, i}& /@
      allλs[i]] /@ (2^Range[Log[2, p - 1], 2, -1]), 1];
    (* do the work *) solStep @@ #& /@ stepArgs; solList]
```

Now, let us calculate the two simple cases $p = 3$ and $p = 5$ as a warm up.

```
In[19]:= (Λ[1, 2] //. GaussSolve[3, Λ])/2
```

$$Out[19]= \ -\frac{1}{2}$$

In[20]:= **(Λ[1, 2] //. GaussSolve[5, Λ])/2 // Expand**

Out[20]= $-\dfrac{1}{4} + \dfrac{\sqrt{5}}{4}$

The results agree with the well-known expressions for $\cos(2\pi/3)$ and $\cos(2\pi/5)$. Here is the list of the values of the periods for $p = 17$.

In[21]:= **(list17 = GaussSolve[17, Λ]) // InputForm**

Out[21]//InputForm= {Λ[1, 16] -> -1, Λ[1, 8] -> Λ[1, 16]/2 + Sqrt[4 + Λ[1, 16]^2/4],
   Λ[3, 8] -> Λ[1, 16]/2 - Sqrt[4 + Λ[1, 16]^2/4], Λ[1, 4] -> Λ[1, 8]/2 +
   Sqrt[1 + Λ[1, 8]^2/4],
   Λ[9, 4] -> Λ[1, 8]/2 - Sqrt[1 + Λ[1, 8]^2/4], Λ[3, 4] -> Λ[3, 8]/2 +
   Sqrt[1 + Λ[3, 8]^2/4],
   Λ[10, 4] -> Λ[3, 8]/2 - Sqrt[1 + Λ[3, 8]^2/4],
   Λ[1, 2] -> Λ[1, 4]/2 + Sqrt[Λ[1, 4]^2/4 - Λ[3, 4]],
   Λ[13, 2] -> Λ[1, 4]/2 - Sqrt[Λ[1, 4]^2/4 - Λ[3, 4]],
   Λ[9, 2] -> Λ[9, 4]/2 - Sqrt[1 + Λ[1, 8] + Λ[3, 4] + Λ[9, 4]^2/4],
   Λ[15, 2] -> Λ[9, 4]/2 + Sqrt[1 + Λ[1, 8] + Λ[3, 4] + Λ[9, 4]^2/4],
   Λ[3, 2] -> Λ[3, 4]/2 + Sqrt[Λ[1, 4] - Λ[1, 8] + Λ[3, 4]^2/4],
   Λ[5, 2] -> Λ[3, 4]/2 - Sqrt[Λ[1, 4] - Λ[1, 8] + Λ[3, 4]^2/4],
   Λ[10, 2] -> Λ[10, 4]/2 - Sqrt[-Λ[1, 4] + Λ[10, 4]^2/4],
   Λ[11, 2] -> Λ[10, 4]/2 + Sqrt[-Λ[1, 4] + Λ[10, 4]^2/4]}

Here is the final expression for $\cos(\frac{2\pi}{17})$.

In[22]:= **(Λ[1, 2] //. list17)/2 // Expand // Factor**

Out[22]= $\dfrac{1}{16}\left(-1 + \sqrt{17} + \sqrt{2\left(17 - \sqrt{17}\right)} + \right.$

$\left. \sqrt{2\left(34 + 6\sqrt{17} - \sqrt{2\left(17 - \sqrt{17}\right)} + \sqrt{34\left(17 - \sqrt{17}\right)} - 8\sqrt{2\left(17 + \sqrt{17}\right)}\right)} \right)$

We numerically check this result. Because the result *is* 0, we cannot get any significant digit, and so the N::meprec message is issued.

In[23]:= **(% - Cos[2Pi/17]) // SetPrecision[#, 1000]&**

Out[23]= $0. \times 10^{-1000}$

Next is the result for $\cos(2\times 2\pi/17)$. (Because we have eliminated most of the L[j, 2]'s with even j, we make use of $\cos(2j\pi/p) = \cos(2(p-j)\pi/p)$ and use L[15, 2].)

In[24]:= **(Λ[15, 2] //. list17)/2 // Expand // Factor**

Out[24]= $\dfrac{1}{16}\left(-1 + \sqrt{17} - \sqrt{2\left(17 - \sqrt{17}\right)} + \right.$

$\left. \sqrt{2\left(34 + 6\sqrt{17} + \sqrt{2\left(17 - \sqrt{17}\right)} - \sqrt{34\left(17 - \sqrt{17}\right)} + 8\sqrt{2\left(17 + \sqrt{17}\right)}\right)} \right)$

In[25]:= **(% - Cos[2 2Pi/17]) // SetPrecision[#, 1000]&**

Out[25]= $0. \times 10^{-1000}$

Using the powerful function RootReduce we could also prove the last equality symbolically.

In[26]:= **(%% // Simplify // RootReduce) -**
(Together[TrigToExp[Cos[2 2Pi/17]]] // RootReduce)

Out[26]= 0

In[27]:= **Together[TrigToExp[Cos[2 2Pi/17]]]**

Out[27]= $-\frac{1}{2} (-1)^{13/17} (1 + (-1)^{8/17})$

The last value of interest here is $\cos(8 \times 2\pi/17)$.

In[28]:= **(Λ[9, 2] //. list17)/2 // Expand // Factor**

Out[28]= $\frac{1}{16} \left(-1 + \sqrt{17} - \sqrt{2\left(17 - \sqrt{17}\right)} - \right.$

$$\left. \sqrt{2\left(34 + 6\sqrt{17} + \sqrt{2\left(17 - \sqrt{17}\right)} - \sqrt{34\left(17 - \sqrt{17}\right)} + 8\sqrt{2\left(17 + \sqrt{17}\right)}\right)} \right)$$

Here is again a quick numerical check for the last result.

In[29]:= **(% - Cos[8 2Pi/17]) // SetPrecision[#, 1000]&**

Out[29]= $0. \times 10^{-1000}$

Now, as promised in the title of this subsection, we calculate $\cos(2\pi/257)$ [1498★], [643★].

In[30]:= **list257 = GaussSolve[257, Λ];**

We select only those parts that are explicitly needed for the evaluation of $\cos(2\pi/257)$.

```
In[31]:= Flatten[Function[{lhs, rhs},
         (* until we have all needed Λ's *)
         FixedPoint[{#, Complement[Union[Cases[
         (* what is in the rhs *)
         rhs[[#]]& /@ Flatten[Position[lhs, #]& /@
             Last[#]], _Λ, {0, Infinity}]], Flatten[#]]}&,
         (* this we need of course *) {{Λ[1, 2]}},
         SameTest -> (Last[#2] === {}&)]][
                   (* all lhs and rhs from list257 *)
                   First /@ list257, Last /@ list257]];

         solListPiD257 = (list257[[#]]& /@ Flatten[Function[lhs,
             Position[lhs, #]& /@ %][First /@ list257]]);
```

Here is a shortened version of this list of replacement rules necessary to express $\cos(\frac{2\pi}{257})$.

In[33]:= **solListPiD257 // Short[#, 6]&**

Out[33]//Short= $\{\Lambda[1, 2] \rightarrow \frac{1}{2} \Lambda[1, 4] + \sqrt{\frac{1}{4} \Lambda[1, 4]^2 + \Lambda[1, 16] - \Lambda[1, 32] + \Lambda[136, 8] + \Lambda[197, 4]}$,

$\Lambda[1, 4] \rightarrow \frac{1}{2} \Lambda[1, 8] + \sqrt{\frac{1}{4} \Lambda[1, 8]^2 - \Lambda[3, 8] + \Lambda[131, 8] - \Lambda[131, 16]}$,

$\Lambda[1, 16] \rightarrow \frac{1}{2} \Lambda[1, 32] +$

$\sqrt{-\Lambda[1, 32] + \frac{1}{4} \Lambda[1, 32]^2 - \Lambda[1, 128] + 2 \Lambda[3, 32] - 2 \Lambda[3, 64] - \Lambda[9, 32]}$,

$\Lambda[1, 32] \rightarrow \frac{1}{2} \Lambda[1, 64] + \sqrt{5 + 2 \Lambda[1, 64] + \frac{1}{4} \Lambda[1, 64]^2 + \Lambda[1, 128]}$,

$\ll 28 \gg$, $\Lambda[243, 32] \rightarrow \frac{1}{2} \Lambda[3, 64] + \sqrt{4 - \Lambda[1, 128] + 2 \Lambda[3, 64] + \frac{1}{4} \Lambda[3, 64]^2}$,

$\Lambda[27, 64] \rightarrow \frac{1}{2} \Lambda[3, 128] - \sqrt{16 + \frac{1}{4} \Lambda[3, 128]^2}$,

$\Lambda[81, 32] \rightarrow \frac{1}{2} \Lambda[1, 64] - \sqrt{5 + 2 \Lambda[1, 64] + \frac{1}{4} \Lambda[1, 64]^2 + \Lambda[1, 128]}$,

$\Lambda[215, 32] \rightarrow \frac{1}{2} \Lambda[9, 64] - \sqrt{5 - 2 \Lambda[1, 64] + 3 \Lambda[1, 128] + \frac{1}{4} \Lambda[9, 64]^2}\}$

The value for $\cos(\frac{2\pi}{257})$ is now easily obtained, but because of its size, we do not display it here.

In[34]:= **(cos2PiD257 = (Λ[1, 2] //. Dispatch[solListPiD257])/2) // ByteCount**

Out[34]= 1822680

It contains only square roots, but it contains a lot of them.

In[35]:= **Cases[cos2PiD257, Power[_, 1/2],**
       {0, Infinity}, Heads -> True] // Length

Out[35]= 5133

If the reader wants to see all of them, the following code opens a new notebook with the typeset formula for the square root version of $\cos(2\pi/257)$.

```
NotebookPut[Notebook[{Cell[BoxData[
              FormBox[MakeBoxes[#, TraditionalForm]&[cos2PiD257],
                      TraditionalForm]], "Output",
    ShowCellBracket -> False, CellMargins -> {{0, 0}, {5, 5}},
    PageWidth -> Infinity, FontColor -> GrayLevel[1],
    (* allow to see all square roots *) CellHorizontalScrolling -> True]},
    WindowSize -> {Automatic, Fit},
    Background -> RGBColor[0.31, 0., 0.51],
    ScrollingOptions  -> {"HorizontalScrollRange" -> 500000},
    WindowMargins -> {{0, 0}, {Automatic, 10}},
    WindowElements -> {"HorizontalScrollBar"},
    WindowFrameElements -> {"CloseBox"}]]
```

Here is a numerical check of the result.

In[36]:= **(cos2PiD257 – Cos[2Pi/257]) // SetPrecision[#, 1000]&**

Out[36]= $0. \times 10^{-996}$

One could now go on and calculate the following quite large calculation for the denominator 65537.

```
l65537 = GaussSolve[65537, L]
```

It will take around one day on a modern workstation. Here are the first lines of the result (of size 55 MB).
```
{Λ[   1, 65536] -> -1,
 Λ[   1, 32768] -> Λ[1, 65536]/2 + Sqrt[16384 + Λ[1, 65536]^2/4],
 Λ[   3, 32768] -> Λ[1, 65536]/2 - Sqrt[16384 + Λ[1, 65536]^2/4],
 Λ[   1, 16384] -> Λ[1, 32768]/2 - Sqrt[ 4096 + Λ[1, 32768]^2/4],
 Λ[   9, 16384] -> Λ[1, 32768]/2 + Sqrt[ 4096 + Λ[1, 32768]^2/4],
 Λ[   3, 16384] -> Λ[3, 32768]/2 - Sqrt[ 4096 + Λ[3, 32768]^2/4],
 Λ[  27, 16384] -> Λ[3, 32768]/2 + Sqrt[ 4096 + Λ[3, 32768]^2/4],
 Λ[   1,  8192] -> Λ[1, 16384]/2 - Sqrt[ 1040 + 32 Λ[1, 16384] +
                   Λ[1, 16384]^2/4 + 16 Λ[1, 32768]],
 Λ[  81,  8192] -> Λ[1, 16384]/2 + Sqrt[1040 + 32 Λ[1, 16384] +
                   Λ[1, 16384]^2/4 + 16 Λ[1, 32768]],
 Λ[   9,  8192] -> Λ[9, 16384]/2 + Sqrt[1040 - 32 Λ[1, 16384] +
                   48 Λ[1, 32768] + Λ[9, 16384]^2/4],
 Λ[729,  8192] -> Λ[9, 16384]/2 - Sqrt[1040 - 32 Λ[1, 16384] +
                   48 Λ[1, 32768] + Λ[9, 16384]^2/4],
 Λ[   3,  8192] -> Λ[3, 16384]/2 + Sqrt[1024 - 16 Λ[1, 32768] +
                   32 Λ[3, 16384] + Λ[3, 16384]^2/4]}
```

(Although the above implementation strictly follows Gauss's original work, we could have used more efficient procedures. See [843★].)

Let us briefly discuss the numbers *n* for which the value $\cos(2\pi/n)$ can be expressed in square roots (or geometrically speaking, which *n*-gons can be constructed by ruler and compass [471★], [833★]?).

The above-mentioned number $2^{2^5} - 1 = 4294967295$ is not a prime number; the factors are all Fermat numbers F_j with $j = 0, \ldots, 4$.

In[37]:= **FactorInteger[2^(2^5) - 1]**

Out[37]= {{3, 1}, {5, 1}, {17, 1}, {257, 1}, {65537, 1}}

(It is not by accident that we have the relation $F_n = 2 + F_{n-1} F_{n-2} \cdots F_0$; for factorizations of Fermat numbers, see [1090✱] and [762✱].) By solving some simple linear diophantine equations (how to do this in *Mathematica* is discussed in [1817✱]), we easily find that $1/(2^{2^5} - 1)$ can be expressed as the following sum.

In[38]:= **1/(3 5 17 257 65537) -**
         (29129863/3 - 58259726/5 + 33291272/17 - 4194176/257 + 32768/65537)

Out[38]= 0

In general, it is possible to calculate $\cos(2\pi/n)$ in square roots if n is a product q of prime Fermat numbers that contains no factor more than once or if $n = q\,2^j$ (j is a nonnegative integer).

For an explicit ruler and compass construction of the regular 257-gon, see [763✱].

It should be noted that the above approach could be easily generalized to the case of splitting one period into more than two periods. Then, the resulting equations obeyed by the periods are no longer quadratic, but of the order of the number of newly generated periods. For more details on this subject, see [110✱], [485✱], [513✱], [1091✱], [1301✱], [1703✱], [1291✱], [891✱], [379✱], [1672✱], [1732✱], [962✱], [668✱], [1340✱], [1837✱], and [1836✱]. See [133✱], [953✱], [313✱], [1794✱], [1069✱], and [1684✱] for the explicit geometrical construction of the *n*-gon.

In[39]:= **Σ (* session summary *) TMGBs`PrintSessionSummary[]**

○ **Session Summary:** (Evaluated with *Mathematica* 5.1)

Inputs evaluated:	38
CPU time used:	4 s
Wall clock time elapsed:	5 s
Max. memory used:	9 MB
Variables in use:	101

■ 1.10.3 Implicitization of a Trefoil Knot

In this subsection, we will deal with a larger polynomial calculation, the implicitization of a trefoil knot. Implicitization of a parametrically defined surface is, in general, a very useful operation because both the parametric and implicit representations have advantages (see [116✱], [1605✱], [1078✱], and [1422✱]). The parametric version is easy to plot, and the implicit one allows an easy decision on which side of the surface a given point lies or if a point lies on the surface. In Chapter 3 of the Graphics volume [1751✱], we discussed and constructed several polynomial surfaces, but not a knotted one. Using the technique described there of gluing together simple implicitly defined objects (like ellipsoids, spheres, tori, cones, tubes) appropriately spaced and oriented, we could generate knotted objects.

But we consider this approach cheating, and we will construct a clean "knot tube" here.

Using the function `TubeFunctionalShort` from Chapter 2 of the Graphics volume [1751✱], here is a picture of the trefoil knot under consideration.

```
In[1]:= TubeFunctionalShort[
        curve_List          (* parametric representation     *),
        cp_                 (* curve parameter               *),
        rr_                 (* tube radius                   *),
        cpInt_List          (* domain of the curve parameter *),
        ppq_                (* PlotPoints cross section       *),
        ppl_                (* PlotPoints transversal         *),
        opts___             (* possible options for the plot *)] :=
   Show[Graphics3D[Drop[MapThread[Polygon[{#1, #2, #3, #4}]&,
    {#[[1]], #[[2]], RotateRight[#[[3]]], #[[3]]}&[
     {#, RotateRight[#], Transpose[RotateRight[
      Transpose[#]]]}&[ Function[u1n,
    MapThread[Map[Function[temp, temp + #1], #2]&,
      {Function[cp, Evaluate[N[curve]]] /@ u1n,
       Partition[(Plus @@ #)& /@ Distribute[{
        Transpose[Function[{d1n, d2n},
        {(#/Sqrt[#.#])& /@ MapThread[(#2/(#1.#1) -
         (#1.#2) #1/(#1.#1)^2)&, {d1n, d2n}],
    MapThread[(Function[{t1, t2, t3, k1, k2, k3},
          {t2 k3 - t3 k2, t3 k1 - t1 k3, t1 k2 - t2 k1}] @@
           Flatten[{##}]&), {(#/Sqrt[#.#])& /@
       d1n, (#/Sqrt[#.#])& /@ MapThread[(#2/(#1.#1) -
        (#1.#2) #1/(#1.#1)^2)&, {d1n, d2n}]}]}][
     Function[cp, Evaluate[N[D[curve, cp]]]] /@ u1n,
     Function[cp, Evaluate[N[D[curve, {cp, 2}]]]] /@ u1n]],
       rr N[{Cos[#], Sin[#]}& /@
      N[Range[0, 2Pi(1 - 1/ppq), 2Pi/ppq]]]},
            List, List, List, Times], {ppq}]]]][
        N[Range[cpInt[[1]], cpInt[[2]],
        -Subtract @@ cpInt/ppl]]]]], 2], 1]], opts]
```
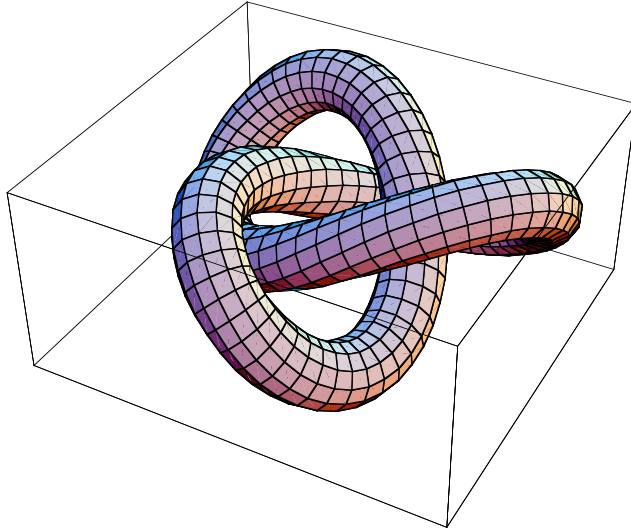
```
In[2]:= TubeFunctionalShort[
          {Sin[t] + 2 Sin[2 t], Cos[t] - 2 Cos[2 t], Sin[3 t]}
                             (* parametric representation    *),
          t                  (* curve parameter              *),
          1/2                (* tube radius                  *),
          {0, 2Pi}           (* domain of the curve parameter *),
          12                 (* PlotPoints cross section      *),
          100                (* PlotPoints transversal        *)]
```


Out[2]= ▪ Graphics3D ▪

Our trefoil knot is a tube of radius 1/2 along the space curve parametrized by

$$\{c_x(t),\, c_y(t),\, c_z(t)\} = \{\sin(t) + 2\sin(2\,t),\, \cos(t) - 2\cos(2\,t),\, \sin(3\,t)\}$$

(a so-called canal surface [195★]). Our goal is to find a function $f(x, y, z)$, such that the tuboidal surface shown last is given by $f(x, y, z) = 0$. To make use of *Mathematica*'s built-in function to construct such an implicit representation, we calculate a polynomial $p(x, y, z)$ describing the above curve.

A first attempt could be to explicitly write down the parametric representation $\{x(s, t), y(s, t), z(s, t)\}$ of the trefoil knot and then try to eliminate the two parametric variables s and t. Because of the normalization conditions on the normal and the binormal vectors to ensure the tube radius of $1/2$, this parametrization is quite a complicated function, involving square roots of polynomials in trigonometric functions of s and t. Therefore, there seems little chance to eliminate s and t from these expressions. So, we choose another approach based on the following two equations [1306★]:

$$|\{x,\, y,\, z\} - \{c_x(t),\, c_y(t),\, c_z(t)\}|^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\left\{\frac{dc_x(t)}{dt},\, \frac{dc_y(t)}{dt},\, \frac{dc_z(t)}{dt}\right\} \cdot (\{x,\, y,\, z\} - \{c_x(t),\, c_y(t),\, c_z(t)\}) = 0.$$

The meaning of these two equations is the following: The first equation expresses the condition that every point $\{x, y, z\}$ of the tube is an element of a sphere of radius $1/2$ around a point $\{c_x(t), c_y(t), c_z(t)\}$ of the space curve. The second equation ensures that the vector from the space curve point $\{c_x(t), c_y(t), c_z(t)\}$ to the point $\{x, y, z\}$ of the tube is always perpendicular to the tangent at the space curve, so that the tube's radius is constant in the perpendicular cross section. This means, these two equations describe exactly the tube we want to implicitize.

The advantage of this description is that now we have only one parameter, the curve parameter t, to eliminate. We do this by

using rational parametrization of the trigonometric functions to transform the problem into a polynomial one. If this is done, we can use known elimination algorithms to eliminate the curve parameter. The resulting polynomial then contains the implicit representation we are looking for as some factor. (For the inverse problem—the derivation of a parametrized description from an implicit description, see [1582⋆].)

There are basically two ways to implicitize equations of the above type [1080⋆], [1255⋆]. The first possibility is to add the equation $\sin(t)^2 + \cos(t)^2 - 1 = 0$ to the equations under consideration and then to eliminate $\sin(x)$ and $\cos(x)$ from the resulting set of polynomial equations. The second possibility is to use the new variable $s = \tan(t/2)$. In this case, we keep the number of variables, but introduce fractions of polynomials. After clearing denominators (we will not care here about the zeros of the denominators; for how to take these single points into account properly, see [418⋆], [437⋆]), we eliminate s from the resulting equations. In the following, we choose the second way. Here, we demonstrate this method for calculating the implicit representation of a torus, using the built-in function `Eliminate` to eliminate the one variable `s`.

```
In[3]:= With[{c = {R Sin[t], R Cos[t], 0} (* the spine curve of a torus *)},
        Factor @ (Subtract @@ (Eliminate[# == 0& /@
        (({#.#&[{x, y, z} - c] - r^2, D[c, t].({x, y, z} - c)} /.
            {Cos[t] -> (1 - s^2)/(1 + s^2), Sin[t] -> 2 s/(1 + s^2)})), {s}]))]
```

Out[3]= $r^4 - 2\, r^2\, R^2 + R^4 - 2\, r^2\, x^2 - 2\, R^2\, x^2 + x^4 - 2\, r^2\, y^2 -$
  $2\, R^2\, y^2 + 2\, x^2\, y^2 + y^4 - 2\, r^2\, z^2 + 2\, R^2\, z^2 + 2\, x^2\, z^2 + 2\, y^2\, z^2 + z^4$

The last output is the implicit representation of the torus.

For our trefoil knot, the two equations derived above are given by the following inputs.

```
In[4]:= c = {Sin[t] + 2 Sin[2t], Cos[t] - 2 Cos[2t], Sin[3t]}
```

Out[4]= $\{Sin[t] + 2\, Sin[2\, t], Cos[t] - 2\, Cos[2\, t], Sin[3\, t]\}$

```
In[5]:= {#.#&[{x, y, z} - c] - (1/2)^2, D[c, t].({x, y, z} - c)}
```

Out[5]= $\Big\{ -\dfrac{1}{4} + (y - Cos[t] + 2\, Cos[2\, t])^2 + (x - Sin[t] - 2\, Sin[2\, t])^2 + (z - Sin[3\, t])^2,$
  $(Cos[t] + 4\, Cos[2\, t])\, (x - Sin[t] - 2\, Sin[2\, t]) +$
  $(y - Cos[t] + 2\, Cos[2\, t])\, (-Sin[t] + 4\, Sin[2\, t]) + 3\, Cos[3\, t]\, (z - Sin[3\, t]) \Big\}$

We express all trigonometric functions of multiple arguments in $\sin(t)$ and $\cos(t)$.

```
In[6]:= % // TrigExpand
```

Out[6]= $\Big\{ \dfrac{21}{4} + x^2 + y^2 + z^2 - 2\, y\, Cos[t] + 4\, y\, Cos[t]^2 - 4\, Cos[t]^3 - \dfrac{Cos[t]^6}{2} - 2\, x\, Sin[t] -$
  $8\, x\, Cos[t]\, Sin[t] - 6\, z\, Cos[t]^2\, Sin[t] - 4\, y\, Sin[t]^2 + 12\, Cos[t]\, Sin[t]^2 +$
  $\dfrac{15}{2}\, Cos[t]^4\, Sin[t]^2 + 2\, z\, Sin[t]^3 - \dfrac{15}{2}\, Cos[t]^2\, Sin[t]^4 + \dfrac{Sin[t]^6}{2},$
  $x\, Cos[t] + 4\, x\, Cos[t]^2 + 3\, z\, Cos[t]^3 - y\, Sin[t] + 8\, y\, Cos[t]\, Sin[t] -$
  $18\, Cos[t]^2\, Sin[t] - 9\, Cos[t]^5\, Sin[t] - 4\, x\, Sin[t]^2 -$
  $9\, z\, Cos[t]\, Sin[t]^2 + 6\, Sin[t]^3 + 30\, Cos[t]^3\, Sin[t]^3 - 9\, Cos[t]\, Sin[t]^5 \Big\}$

Using rational parametrizations for $\sin(t)$ and $\cos(t)$, we get the following two polynomials in s after writing both equations over a common denominator.

In[7]:= `% /. {Cos[t] -> (1 - s^2)/(1 + s^2),`
        `Sin[t] -> 2 s/(1 + s^2)}`

Out[7]= $\left\{ \dfrac{21}{4} + \dfrac{32\,s^6}{(1+s^2)^6} - \dfrac{120\,s^4\,(1-s^2)^2}{(1+s^2)^6} + \dfrac{30\,s^2\,(1-s^2)^4}{(1+s^2)^6} - \dfrac{(1-s^2)^6}{2\,(1+s^2)^6} + \right.$

$\dfrac{48\,s^2\,(1-s^2)}{(1+s^2)^3} - \dfrac{4\,(1-s^2)^3}{(1+s^2)^3} - \dfrac{16\,s\,(1-s^2)\,x}{(1+s^2)^2} - \dfrac{4\,s\,x}{1+s^2} + x^2 - \dfrac{16\,s^2\,y}{(1+s^2)^2} +$

$\dfrac{4\,(1-s^2)^2\,y}{(1+s^2)^2} - \dfrac{2\,(1-s^2)\,y}{1+s^2} + y^2 + \dfrac{16\,s^3\,z}{(1+s^2)^3} - \dfrac{12\,s\,(1-s^2)^2\,z}{(1+s^2)^3} + z^2,$

$-\dfrac{288\,s^5\,(1-s^2)}{(1+s^2)^6} + \dfrac{240\,s^3\,(1-s^2)^3}{(1+s^2)^6} - \dfrac{18\,s\,(1-s^2)^5}{(1+s^2)^6} + \dfrac{48\,s^3}{(1+s^2)^3} -$

$\dfrac{36\,s\,(1-s^2)^2}{(1+s^2)^3} - \dfrac{16\,s^2\,x}{(1+s^2)^2} + \dfrac{4\,(1-s^2)^2\,x}{(1+s^2)^2} + \dfrac{(1-s^2)\,x}{1+s^2} +$

$\left. \dfrac{16\,s\,(1-s^2)\,y}{(1+s^2)^2} - \dfrac{2\,s\,y}{1+s^2} - \dfrac{36\,s^2\,(1-s^2)\,z}{(1+s^2)^3} + \dfrac{3\,(1-s^2)^3\,z}{(1+s^2)^3} \right\}$

In[8]:= `{p1, p2} = Numerator /@ Together /@ %`

Out[8]= $\{3 + 450\,s^2 - 243\,s^4 + 2268\,s^6 - 1107\,s^8 + 66\,s^{10} + 35\,s^{12} - 80\,s\,x -$
$272\,s^3\,x - 288\,s^5\,x - 32\,s^7\,x + 112\,s^9\,x + 48\,s^{11}\,x + 4\,x^2 + 24\,s^2\,x^2 + 60\,s^4\,x^2 +$
$80\,s^6\,x^2 + 60\,s^8\,x^2 + 24\,s^{10}\,x^2 + 4\,s^{12}\,x^2 + 8\,y - 64\,s^2\,y - 312\,s^4\,y - 448\,s^6\,y -$
$232\,s^8\,y + 24\,s^{12}\,y + 4\,y^2 + 24\,s^2\,y^2 + 60\,s^4\,y^2 + 80\,s^6\,y^2 + 60\,s^8\,y^2 +$
$24\,s^{10}\,y^2 + 4\,s^{12}\,y^2 - 48\,s\,z + 16\,s^3\,z + 288\,s^5\,z + 288\,s^7\,z + 16\,s^9\,z -$
$48\,s^{11}\,z + 4\,z^2 + 24\,s^2\,z^2 + 60\,s^4\,z^2 + 80\,s^6\,z^2 + 60\,s^8\,z^2 + 24\,s^{10}\,z^2 + 4\,s^{12}\,z^2,$
$-54\,s + 342\,s^3 - 972\,s^5 + 1404\,s^7 - 318\,s^9 - 18\,s^{11} + 5\,x - 4\,s^2\,x - 63\,s^4\,x -$
$112\,s^6\,x - 73\,s^8\,x - 12\,s^{10}\,x + 3\,s^{12}\,x + 14\,s\,y + 38\,s^3\,y + 12\,s^5\,y - 52\,s^7\,y -$
$58\,s^9\,y - 18\,s^{11}\,y + 3\,z - 36\,s^2\,z - 81\,s^4\,z + 81\,s^8\,z + 36\,s^{10}\,z - 3\,s^{12}\,z\}$

We now have to calculate the resultant of two polynomials of degree 12 with symbolic coefficients. This is potentially quite a large calculation. There are various possibilities, for instance, the subresultant method [1266✹], [253✹], [1285✹], [390✹], [391✹], and [1164✹] (which requires calculating greatest common divisors of multivariate polynomials, which we are not going to do here to avoid very complicated polynomial work), using Sylvester's determinant formulation (which, as discussed in Section 1.2, results in calculating the determinant of a not sparse 24×24 matrix with symbolic entries, a very stunning task, too), or making an ansatz for the resulting polynomial in *x*, *y*, and *z* and determining the unknown coefficients (for instance, via interpolation [1208✹], [1209✹], [704✹]) or Cayley's method [990✹], [1839✹], [1840✹], [779✹], and [713✹] for the resultant calculation (and use Laplace's expansion [1120✹], [1793✹] recursively of the determinant). Here we use the `Method -> Modular` option setting of the built-in function `Resultant`.

In[9]:= `(treFoilKnotPoly[x_, y_, z_] =`
        `Factor[Resultant[p1, p2, s, Method -> Modular]]); // Timing`

Out[9]= $\{48.49\ \text{Second, Null}\}$

In[10]:= `treFoilKnotPoly[x, y, z] // Short[#, 16]&`

Out[10]//Short= $1329227995784915872903807060280344576$

$(216476419441409357625 - 7014002132627225648640\,x^2 +$
$2557575896813793543168\,x^4 + 16372145598881992180\,x^6 -$
$35920473838281441792\,x^8 + 1110145266874574434\,x^{10} -$
$4896337446674124800\,x^{12} - 836052734165745664\,x^{14} +$
$292894441143336960\,x^{16} + 2947572217413632\,x^{18} - 5608909354041344\,x^{20} +$
$379637528002560\,x^{22} + 8916100448256\,x^{24} - 17950247748010258851456\,x^2\,y +$
$16014563930164085155584\,x^4\,y - 4092306021947893469184\,x^6\,y +$
$107486645236502704128\,x^8\,y + 101077487797146107904\,x^{10}\,y -$
$39569792051317112832\,x^{12}\,y + 2121291115181899776\,x^{14}\,y +$
$1455106505387802624\,x^{16}\,y - 213048086451191808\,x^{18}\,y +$
$5501130696032256\,x^{20}\,y + 507227047723008\,x^{22}\,y - 7014002132627225648640\,y^2 +$
$\ll\!1634\!\gg + 243468982907043840\,x^2\,y^6\,z^{14} + 427972821516288\,y^7\,z^{14} +$
$60867245726760960\,y^8\,z^{14} - 570630428688384\,y^9\,z^{14} +$
$959377691847426048\,x^3\,z^{15} + 248224236479447040\,x^5\,z^{15} +$
$6847565144260608\,x^5\,y\,z^{15} - 2878133075542278144\,x\,y^2\,z^{15} -$
$496448472958894080\,x^3\,y^2\,z^{15} - 22825217147535360\,x^3\,y^3\,z^{15} -$
$744672709438341120\,x\,y^4\,z^{15} + 6847565144260608\,x\,y^5\,z^{15} +$
$1236413481360556032\,z^{16} + 252979490051850240\,x^2\,z^{16} +$
$9130086859014144\,x^6\,z^{16} - 337813213783523328\,x^2\,y\,z^{16} +$
$252979490051850240\,y^2\,z^{16} + 27390260577042432\,x^4\,y^2\,z^{16} +$
$112604404594507776\,y^3\,z^{16} + 27390260577042432\,x^2\,y^4\,z^{16} +$
$9130086859014144\,y^6\,z^{16} + 36520347436056576\,x^3\,z^{17} -$
$109561042308169728\,x\,y^2\,z^{17} + 36520347436056576\,z^{18})$

The result is a large polynomial. It is of total degree 24, has 1212 terms and coefficients with up to 23 digits.

In[11]:= `(* remove large factored-out integer prefactor *)`
`treFoilKnotPoly[x_, y_, z_] = treFoilKnotPoly[x, y, z] [[-1]];`

In[13]:= `{(* number of terms *) Length @ treFoilKnotPoly[x, y, z],`
` (* total degree *) Exponent[treFoilKnotPoly[τ, τ, τ], τ],`
` (* largest coefficient *)`
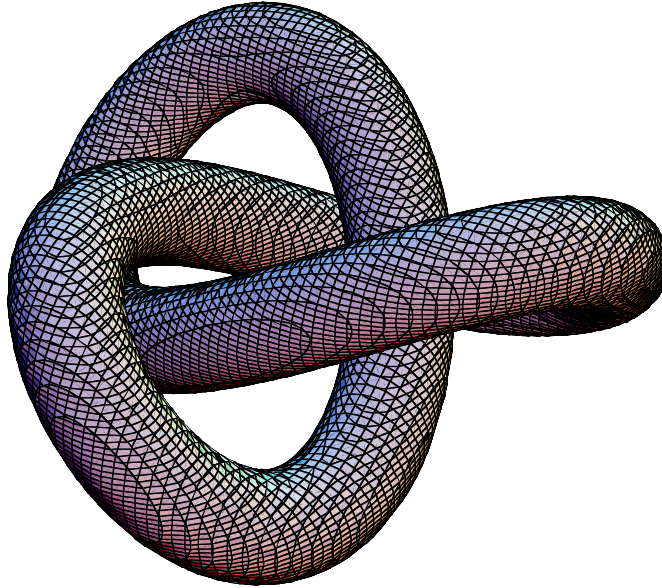` Max[Abs[Cases[treFoilKnotPoly[x, y, z], _Integer, {-1}]]] // N}`

Out[13]= $\{1212, 24, 3.97887 \times 10^{22}\}$

We now use the just-calculated implicit formula to make a graphic of the knot.

In[14]:=
```
Needs["Graphics`ContourPlot3D`"]

ContourPlot3D[Evaluate[treFoilKnotPoly[x, y, z]],
              {x, -4, 4}, {y, -4, 4}, {z, -3/2, 3/2},
              MaxRecursion -> 1, Boxed -> False,
              PlotPoints -> {{21, 5}, {21, 5}, {16, 5}}]
```
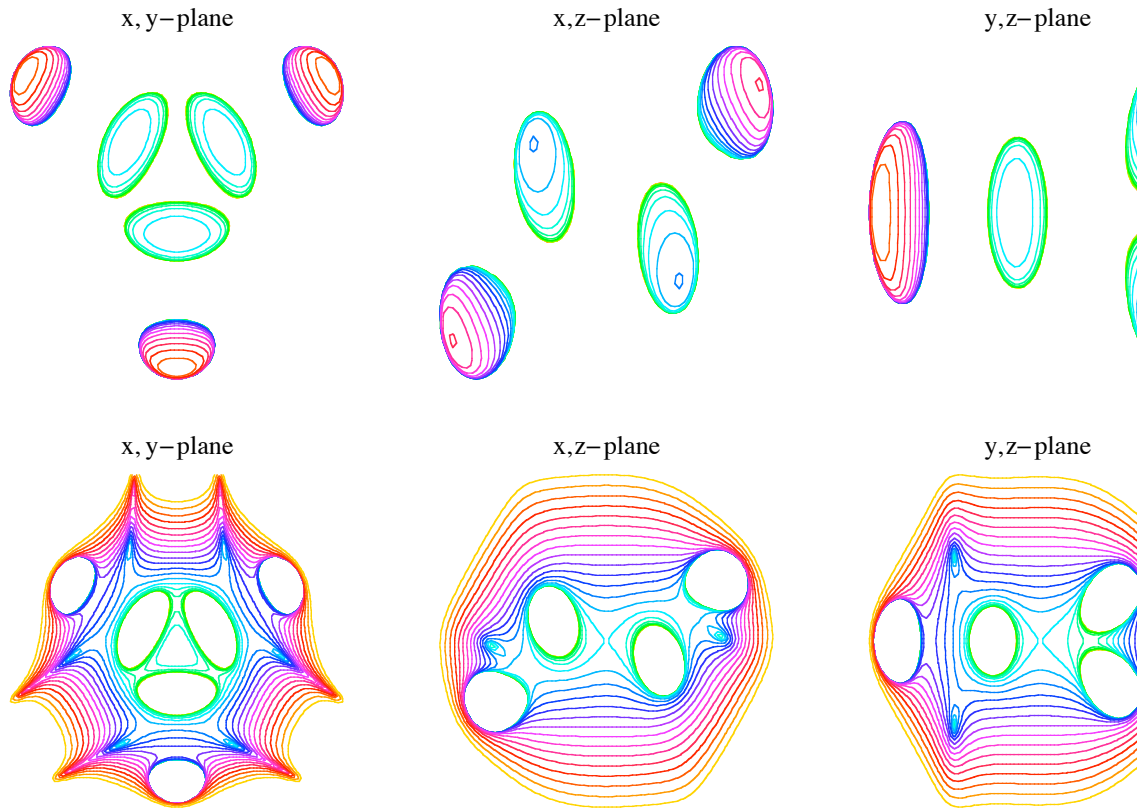

Out[15]= **- Graphics3D -**

We could now continue and slowly morph the knot into a ball (similar to Section 3.3 of the Graphics volume) [1760✶]. Here are the coordinate plane cross sections of such a morphing. We leave it to the reader to implement the corresponding 3D animation.

In[16]:=
```
Function[sign, (* add and subtract constant *)
Show[GraphicsArray[Show /@ Transpose[Table[
ContourPlot[Evaluate[treFoilKnotPoly[#1, #2, #3] +
                     (* build and cut connections *) sign 10^ε],
              {#4, -5, 5}, {#5, -4, 4}, PlotPoints -> 120,
              Contours -> {0}, ContourShading -> False,
              ContourStyle -> {Hue[0.8 (ε - 55)/6]},
              DisplayFunction -> Identity, Frame -> False,
              PlotLabel -> #6 <> "-plane"]& @@@
         (* the 3 coordinate plane data *)
         {{x, y, 0, x, y, "x,y"}, {x, 0, z, x, z, "x,z"},
          {0, y, z, y, z, "y,z"}}, {ε, 18, 26, 1/3}]]]]] /@ {+1, -1}
```

x, y−plane   x, z−plane   y, z−plane

x, y−plane   x, z−plane   y, z−plane

Out[16]= {■ GraphicsArray ■, ■ GraphicsArray ■}

And here are two 3D plots of the resulting surfaces. By adding a constant to the polynomial, we squeeze the tube and by subtracting a constant, we thicken the tube.

```
In[17]:=  Show[GraphicsArray[
          (* show squeezed and fattened version *)
          Graphics3D[{EdgeForm[], Cases[
           ContourPlot3D[Evaluate[treFoilKnotPoly[x, y, z] + #],
                         {x, -5, 5}, {y, -5, 5}, {z, -2, 2}, Boxed -> False,
                         MaxRecursion -> 1, DisplayFunction -> Identity,
                         PlotPoints -> {{21, 6}, {21, 6}, {13, 6}}],
                                         _Polygon, Infinity] /. (* cut vertices off *
                      Polygon[l_] :> Polygon[Plus @@@
                                             Partition[Append[l, l[[1]]], 2, 1]/2]}
           Boxed -> False]& /@ (* two constant values *) {8 10^21, -10^23}]]
```



```
Out[17]=  - GraphicsArray -
```

In a similar manner, one can implicitize many other surfaces, when their parametrization is in terms of trigonometric or hyperbolic functions, for instance, the Klein bottle from Section 2.2.1 of the Graphics volume [1751★]. Here is their implicit form together with the code for making a picture of the resulting polynomial. (For the implicitization of a "realistic looking" Klein bottle, see [1749★].)

```
Needs["Graphics`ContourPlot3D`"]

Clear[x, y, z, r, φ]

Show[Graphics3D[
(* convert back from polar coordinates to Cartesian coordinates *)
Apply[{#1 Cos[#2], #1 Sin[#2], #3}&,
Cases[ContourPlot3D[Evaluate[
768 x^4 - 1024 x^5 - 128 x^6 + 512 x^7 - 80 x^8 - 64 x^9 + 16 x^10 +
144 x^2 y^2 - 768 x^3 y^2 - 136 x^4 y^2 + 896 x^5 y^2 - 183 x^6 y^2 -
176 x^7 y^2 + 52 x^8 y^2 + 400 y^4 + 256 x y^4 - 912 x^2 y^4 +
256 x^3 y^4 + 315 x^4 y^4 - 144 x^5 y^4 - 16 x^6 y^4 + 4 x^8 y^4 -
904 y^6 - 128 x y^6 + 859 x^2 y^6 - 16 x^3 y^6 - 200 x^4 y^6 +
16 x^6 y^6 + 441 y^8 + 16 x y^8 - 224 x^2 y^8 + 24 x^4 y^8 - 76 y^10 +
16 x^2 y^10 + 4 y^12 - 2784 x^3 y z + 4112 x^4 y z - 968 x^5 y z -
836 x^6 y z + 416 x^7 y z - 48 x^8 y z + 1312 x y^3 z + 2976 x^2 y^3 z -
5008 x^3 y^3 z - 12 x^4 y^3 z + 2016 x^5 y^3 z - 616 x^6 y^3 z -
64 x^7 y^3 z + 32 x^8 y^3 z - 1136 y^5 z - 4040 x y^5 z +
2484 x^2 y^5 z + 2784 x^3 y^5 z - 1560 x^4 y^5 z - 192 x^5 y^5 z +
128 x^6 y^5 z + 1660 y^7 z + 1184 x y^7 z - 1464 x^2 y^7 z -
192 x^3 y^7 z + 192 x^4 y^7 z - 472 y^9 z - 64 x y^9 z + 128 x^2 y^9 z +
32 y^11 z - 752 x^4 z^2 + 1808 x^5 z^2 - 1468 x^6 z^2 + 512 x^7 z^2 -
64 x^8 z^2 + 6280 x^2 y^2 z^2 - 5728 x^3 y^2 z^2 - 4066 x^4 y^2 z^2 +
5088 x^5 y^2 z^2 - 820 x^6 y^2 z^2 - 384 x^7 y^2 z^2 + 96 x^8 y^2 z^2 -
136 y^4 z^2 - 7536 x y^4 z^2 + 112 x^2 y^4 z^2 + 8640 x^3 y^4 z^2 -
2652 x^4 y^4 z^2 - 1152 x^5 y^4 z^2 + 400 x^6 y^4 z^2 + 2710 y^6 z^2 +
4064 x y^6 z^2 - 3100 x^2 y^6 z^2 - 1152 x^3 y^6 z^2 + 624 x^4 y^6 z^2 -
1204 y^8 z^2 - 384 x y^8 z^2 + 432 x^2 y^8 z^2 + 112 y^10 z^2 +
3896 x^3 y z^3 - 7108 x^4 y z^3 + 3072 x^5 y z^3 + 768 x^6 y z^3 -
768 x^7 y z^3 + 128 x^8 y z^3 - 3272 x y^3 z^3 - 4936 x^2 y^3 z^3 +
8704 x^3 y^3 z^3 - 80 x^4 y^3 z^3 - 2496 x^5 y^3 z^3 + 608 x^6 y^3 z^3 +
2172 y^5 z^3 + 5632 x y^5 z^3 - 2464 x^2 y^5 z^3 - 2688 x^3 y^5 z^3 +
1056 x^4 y^5 z^3 - 1616 y^7 z^3 - 960 x y^7 z^3 + 800 x^2 y^7 z^3 +
224 y^9 z^3 + 752 x^4 z^4 - 1792 x^5 z^4 + 1472 x^6 z^4 - 512 x^7 z^4 +
64 x^8 z^4 - 3031 x^2 y^2 z^4 + 1936 x^3 y^2 z^4 + 2700 x^4 y^2 z^4 -
2304 x^5 y^2 z^4 + 448 x^6 y^2 z^4 + 697 y^4 z^4 + 3728 x y^4 z^4 +
24 x^2 y^4 z^4 - 3072 x^3 y^4 z^4 + 984 x^4 y^4 z^4 - 1204 y^6 z^4 -
1280 x y^6 z^4 + 880 x^2 y^6 z^4 + 280 y^8 z^4 - 800 x^3 y z^5 +
1488 x^4 y z^5 - 768 x^5 y z^5 + 128 x^6 y z^5 + 992 x y^3 z^5 +
1016 x^2 y^3 z^5 - 1728 x^3 y^3 z^5 + 480 x^4 y^3 z^5 - 472 y^5 z^5 -
960 x y^5 z^5 + 576 x^2 y^5 z^5 + 224 y^7 z^5 + 16 x^4 z^6 +
388 x^2 y^2 z^6 - 384 x^3 y^2 z^6 + 96 x^4 y^2 z^6 - 76 y^4 z^6 -
384 x y^4 z^6 + 208 x^2 y^4 z^6 + 112 y^6 z^6 - 64 x y^3 z^7 +
32 x^2 y^3 z^7 + 32 y^5 z^7 + 4 y^4 z^8 /.
 (* to polar coordinates *)  {x -> r Cos[φ], y -> r Sin[φ]}],
{r, 0.6, 3.3}, {φ, 0, 2Pi}, {z, -1.3, 1.3}, PlotPoints -> {18, 40, 24},
MaxRecursion -> 0, DisplayFunction -> Identity], _Polygon, Infinity], {-2}]]]
```
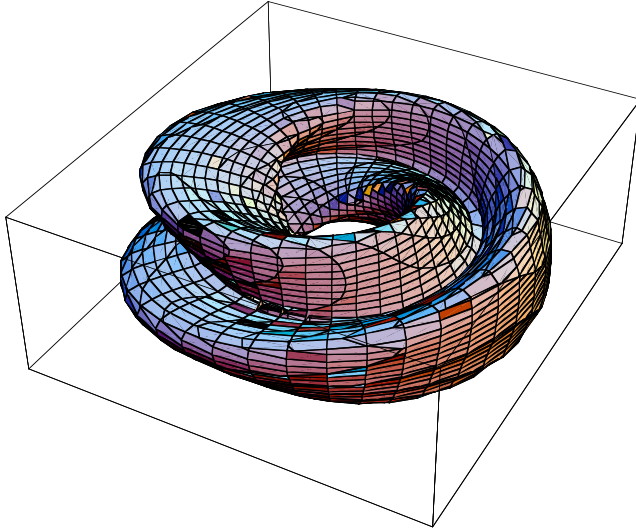
For more on the subject of implicitization of surfaces, see [1208★], [354★], and [1603★] and references cited therein.

We end with another implicit surface originating from a trefoil knot. Starting with a parametrized space curve $c(t)$, we construct the parametrized surface $(c(t + \alpha/2) + c(t + \alpha/2))/\alpha$ (the average of two symmetrically located points with respect to t). The following code calculates the implicit form of this surface for the trefoil knot. We use the function `Resultant` to eliminate the parametrization variables. For brevity, we express the resulting surface in cylindrical coordinates.

```
(* a function to convert from trigonometric to polynomial variables *)
F[expr_] := Numerator[Together[TrigToExp[expr] /.
                                {t -> Log[T]/I, α -> Log[A]/I}]]
(* make algebraic form of average *)
cAv = ((c /. t -> t + α) + (c /. t -> t - α))/2
cAvAlg = F[{x, y, z} - cAv]/{I, 1, I}
(* eliminate parametrization variables *)
res1 = Resultant[cAvAlg[[1]], cAvAlg[[2]], A] // Factor
res2 = Resultant[cAvAlg[[1]], cAvAlg[[3]], A] // Factor
res3 = Resultant[res1[[-1]] /. T -> Sqrt[T2],
                 res2[[-1, 1]] /. T -> Sqrt[T2], T2,
                 Method -> SylvesterMatrix];
(* express implicit form of surface in cylindrical coordinates *)
cAvImpl  = Factor[res3][[3, 1]] /.
                 {x -> r Cos[φ], y -> r Sin[φ]} // FullSimplify
```

```
In[18]:= cAvImpl = r^6 (2 + r) (r - 2) (1 - 44 r^2 + 64 r^4) +
         24 r^4 (-12 - 3 r^2 + 80 r^4) z^2 -
         128 r^2 (-123 + 36 r^2 + 64 r^4) z^4 - 8192 z^6 +
         r^3 (2 z (993 r^4 - 80 r^6 - 4144 z^2 + 8192 z^4 +
         r^2 (84 - 5760 z^2)) Cos[3 φ] + r^3 (-4 + 177 r^2 - 300 r^4 + 64 r^6 -
                         32 (-109 + 48 r^2) z^2) Cos[6 φ] - 64 r^6 z Cos[9 φ] -
         16 (3 r^6 (-4 + r^2) + 2 r^2 (69 - 114 r^2 + 64 r^4) z^2 -
                                       256 (-2 + 3 r^2) z^4) Sin[3 φ] +
         4 r^3 z (157 - 174 r^2 + 512 z^2) Sin[6 φ] - 48 r^6 (-4 + r^2) Sin[9 φ]);

In[19]:= Needs["Graphics`ContourPlot3D`"]
```

```
In[20]:= (* a function for making a hole in a polygon *)
    makeHole[Polygon[l_], f_] :=
    Module[{mp = Plus @@ l/Length[l], ı, ℓ},
      ı = Append[l, First[l]];  ℓ = (mp + f (# - mp))& /@ ı;
      {(* new polygons *) MapThread[Polygon[Join[#1, Reverse[#2]]]&,
                                Partition[#, 2, 1]& /@ {ı, ℓ}]}]
```

The next pair of graphics shows the parametric and the implicit version of this surface. We make use of the threefold rotational symmetry of the surface in the generation of the implicit plot.

```
In[22]:= Show[GraphicsArray[
    Block[{$DisplayFunction = Identity, polysCart,
          R = {{-1, Sqrt[3], 0}, {-Sqrt[3], -1, 0}, {0, 0, 2}}/2.},
    {(* the parametrized 3D plot *)
     ParametricPlot3D[Evaluate[Append[
       ((c /. t -> t + α) + (c /. t -> t - α))/2,
       {EdgeForm[], SurfaceColor[#, #, 3]&[Hue[(t + Pi)/(2Pi)]]}]],
                      {t, -Pi, Pi}, {α, 0, Pi/2}, Axes -> False,
                       PlotPoints -> {64, 32}, BoxRatios -> {1, 1, 0.6},
                       PlotRange -> {{-3, 3}, {-3, 3}, {-1, 1}}] /.
                       p_Polygon :> makeHole[p, 0.76],
     (* the implicit 3D plot; use symmetry *)
    polysCart = Apply[{#1 Cos[#2], #1 Sin[#2], #3}&,
     Cases[(* contour plot in cylindrical coordinates *)
     ContourPlot3D[cAvImpl, {r, 0, 3}, {φ, -Pi/3, Pi/3}, {z, -1, 1},
                      PlotPoints -> {28, 24, 32}, MaxRecursion -> 0],
                       _Polygon, Infinity], {-2}];
     Graphics3D[{EdgeForm[], (* generate all three parts of the surface *)
        {polysCart, Map[  R.#&, polysCart, {-2}],
                    Map[R.R.#&, polysCart, {-2}]}} /.
         p_Polygon :> {SurfaceColor[#, #, 2.4]&[
                        Hue[Sqrt[#.#]&[0.24 Plus @@ p[[1]]]/Length[p[[1]]]]]],
                        makeHole[p, 0.72]}, BoxRatios -> {1, 1, 0.6}]}]]]
```
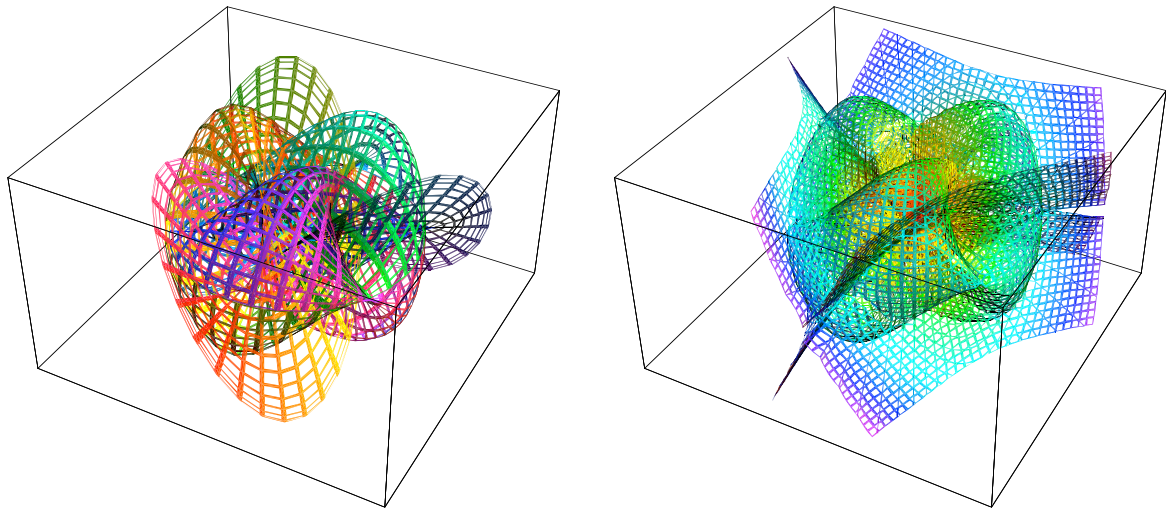


```
Out[22]=  - GraphicsArray -
```

For the volume of such tubes, see [311✶].

In[23]:= **Σ (* session summary *) TMGBs`PrintSessionSummary[]**

○ **Session Summary:** (Evaluated with *Mathematica* 5.1)

Inputs evaluated:	22
CPU time used:	2127 s
Wall clock time elapsed:	2147 s
Max. memory used:	168 MB
Variables in use:	57

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